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April 2017

Discrepancy estimates for point sets and sequences

Doctoral Thesis
to obtain the academic degree of
Doktor der technischen Wissenschaften
in the Doctoral Program
Technische Wissenschaften

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Abstract

The core topic of this thesis goes back to the question on how evenly a deterministic point set or sequence can actually be distributed in the d-dimensional unit cube. The star discrepancy is one of the best known classical measures to quantify this quality. For point sets $\mathcal{P} \subseteq [0,1)^d$, $\#\mathcal{P} = N$, it is denoted by $D_N^*(\mathcal{P})$ and defined as the L^∞-norm of the so-called discrepancy function

$$D_N(\mathcal{P}, \boldsymbol{x}) = \#(\mathcal{P} \cap [0, \boldsymbol{x})) - N\lambda_d([0, \boldsymbol{x})),$$

where $[0, \boldsymbol{x})$ denotes the axis-parallel box anchored at the origin and its endpoints given by the coordinates of $\boldsymbol{x} \in [0,1)^d$, and where λ_d denotes the d-dimensional Lebesgue measure. For sequences $\mathcal{S}$ the star discrepancy $D_N^*(\mathcal{S})$ is defined analogously for the initial segment of N elements.

Two central questions immediately emerge from this definition. First of all, it is of utmost interest to know how well it is possible to deterministically approximate uniform distribution on $[0,1)^d$, and, secondly, one seeks explicit consctruction principles for point sets or sequences for which a low discrepancy is obtained.

The first of these tasks leads to the study of lower bounds for the star discrepancy which hold for all point sets or sequences. As the discrepancy function heavily depends on the choice of $\mathcal{P}$ (or $\mathcal{S}$) it hardly comes as a surprise that the problem of finding the best possible lower bounds for arbitrary such choices has been around for many decades and still remains unsolved for the most part. Henceforward, the first part of this thesis is dedicated to shedding light on the historic development of this field and the evolution of its inherent strategies and techniques, and, additionally, to contributing towards this long standing problem with two recent results.

One of them is concerned with sequences in the unit interval. In this setting, it is known due to Schmidt (1972) that

$$D_N^*(\mathcal{S}) \geq c \cdot \log N$$

for all sequences $\mathcal{S}$, infinitely many N and a constant $c > 0$ independent of N and $\mathcal{S}$. An explicit sequence satisfying an upper bound of essentially

$\log N$ was long known due to Van der Corput (1935). Let us denote by c^* the supremum of all c satisfying this bound. Within this thesis it is demonstrated that $c^* \geq 0.065664679\ldots$ as a result of a joint work of Larcher together with the author (2016), thereby slightly improving earlier results of Larcher (2015) and Béjian (1982).

The second improvement to lower discrepancy bounds that is presented here considers point sets in three dimensions. The exact growth rate of D_N^* for $d \geq 3$ is still merely a subject of speculation. The last major improvements are due to Bilyk and Lacey for $d = 3$ (2008) and to Bilyk, Lacey, and Vagharshakyan for $d \geq 4$ (2008). They could show that there exists a positive number η_d such that

$$D_N^*(\mathcal{P}) \geq c_d \, (\log N)^{(d-1)/2+\eta_d}$$

holds for all N-point sets $\mathcal{P}$ and all N sufficiently large. In 2016 the author published a paper that builds upon their methods and verifies that, for $d = 3$, this bound holds with $\eta_3 = 1/(32+4\sqrt{41})-\epsilon \approx 0.017357\ldots$, for all $\epsilon > 0$. To provide the reader with a clear picture of the sophisticated tools developed and partly rediscovered by Bilyk and Lacey and the strategies that lead to this explicit upper bound, this thesis provides a rigorous proof of the author's result.

With a view to the second main part, i.e. the construction of evenly distributed point sets and sequences, this thesis follows the relatively modern approach of so-called hybrid sequences. These are basically formed by the juxtaposition of sequences from different classes of (classical) low discrepancy sequences. Following a joint effort between Hofer and the author, which has already been submitted for publication, the object that is investigated here is a two-dimensional sequence $(\boldsymbol{z}_k)_{k\geq 0}$, $\boldsymbol{z}_k = (x_k, \{k\alpha\})$, where $\alpha \in (0,1)$ is irrational and $(x_k)_{k\geq 0}$ denotes a digital sequence in the sense of Niederreiter.

By definition, the construction of $(x_k)_{k\geq 0}$ relies on an infinite matrix C with entries in $\{0,1\}$. Two special cases of such sequences were studied by Niederreiter (2009) and by Aistleitner, Hofer, and Larcher (2016). By taking the identity in place of C (so-called Halton–Kronecker sequence) on the one hand, $(\boldsymbol{z}_k)_{k\geq 0}$ is subject to an optimal metric discrepancy bound. Taking C as the identity again, but replacing its first row with an infinite sequence of 1's (related to the so-called evil Kronecker sequence) on the other hand, worsens the metrical behavior of $(\boldsymbol{z}_k)_{k\geq 0}$ significantly. The core issue of this part of the thesis is to investigate what happens in between these two cases. More precisely, it considers Halton–Kronecker sequences, where the first row of C is exchanged by a periodic perturbation $(c_k)_{k\geq 0}$ of blocks of length ℓ of the form $(1,0,\ldots,0)$. The result of these studies are sharp bounds for

the star discrepancy of $(\boldsymbol{z}_k)_{k\geq 0}$ in the case where α has bounded continued fraction coefficients, which surprisingly worsen with decreasing density of 1's in $(c_k)_{k\geq 0}$, as well as tight metric bounds, which are in line with the previously known results, i.e., the exponents within the estimates approach the (supposedly) optimal value for $\ell \to \infty$.

Moreover, this topic reveals tight connections to lacunary trigonometric products of the form $\prod_{j=0}^{r-1} |\cos(2^j \alpha\pi + c_j\pi/2)|$ and sharp general as well as tight metric estimates for these are derived within the corresponding chapter as a side perk.

Kurzfassung

Das fundamentale Thema dieser Dissertation beruht auf der Frage, wie gleichmäßig eine deterministisch erzeugte Punktmenge oder Folge im d-dimensionalen Einheitswürfel verteilt sein kann. Eine der bekanntesten klassischen Kennzahlen zur Quantifizierung dieser Eigenschaft ist die Sterndiskrepanz. Für Punktmengen $\mathcal{P} \subseteq [0,1)^d$, $\#\mathcal{P} = N$, wird diese üblicherweise mit $D_N^*(\mathcal{P})$ bezeichnet und als die L^∞-Norm der so genannten Diskrepanzfunktion

$$D_N(\mathcal{P}, \boldsymbol{x}) = \#\left(\mathcal{P} \cap [0, \boldsymbol{x})\right) - N\lambda_d\left([0, \boldsymbol{x}\right)$$

definiert. Hierbei bezeichnet $[0, \boldsymbol{x})$ jenen achsenparallele Quader, welcher im Ursprung verankert ist und dessen anderen Endpunkte durch die Koordinaten von $\boldsymbol{x} \in [0,1)^d$ bestimmt sind, und λ_d bezeichnet das d-dimensionale Lebesgue Maß. Für Folgen $\mathcal{S}$ lässt sich die Sterndiskrepanz $D_N^*(\mathcal{S})$ für die ersten N Elemente in analoger Weise zu Punktmengen definieren.

Diese Definition wirft sofort zwei zentrale Fragen auf. Erstens ist es von größtem Interesse, wie gut es überhaupt möglich ist, Gleichverteilung auf $[0,1)^d$ mit deterministischen Methoden zu approximieren und, zweitens, ist natürlich viel an konkreten Konstruktionsmethoden von Punktmengen und Folgen mit kleiner Diskrepanz gelegen.

Die erste dieser Fragen führt zur Suche nach unteren Schranken für die Sterndiskrepanz, welche für alle Punktmengen bzw. Folgen gelten. Da aber die Diskrepanzfunktion offenbar stark von der konkreten Wahl von $\mathcal{P}$ bzw. $\mathcal{S}$ abhängt, ist es wenig überraschend, dass die Suche nach den bestmöglichen unteren Schranken für beliebige $\mathcal{P}$ oder $\mathcal{S}$ Mathematiker schon seit Jahrzehnten beschäftigt und zum größten Teil noch immer ungelöst ist. Deshalb wird im ersten Teil dieser Dissertation versucht, Licht auf die historische Entwicklung dieses Problems, sowie auf die Entwicklung der Strategien und Techniken zur Lösung von diesem zu werfen. Darüber hinaus werden zwei aktuelle Beiträge des Autors zu diesem Thema präsentiert.

Der erste Beitrag betrifft Folgen im Einheitsintervall. In diesem Setting gelang es W.M. Schmidt 1972 zu beweisen, dass für alle Folgen $\mathcal{S}$ und unendlich vielen N eine von N und $\mathcal{S}$ unabhängige Konstante $c > 0$ existiert,

sodass

$$D_N^*(\mathcal{S}) \geq c \cdot \log N.$$

Eine explizite Folge, welche im Wesentlichen eine obere Diskrepanzschranke von $\log N$ aufweist, war durch Van der Corput bereits 1935 bekannt. Sei nun c^* das Supremum über alle Konstanten c, die diese Schranke erfüllen. Beruhend auf einer gemeinsamen Arbeit des Authors mit Larcher (2016) wird innerhalb dieser Dissertation das Resultat $c^* \geq 0.065664679\ldots$ hergeleitet, welches frühherere Erbebnisse von Larcher (2015) und Béjian (1982) etwas verbessert.

Der zweite Beitrag bezüglich Verbesserungen von unteren Diskrepanzschranken, der hier diskutiert wird, betrifft Punktmengen im Einheitswürfel. In Dimension $d \geq 3$ ist die exakte Wachstumsrate der Sterndiskrepanz in N noch immer ungelöst und deshalb eine rein spekulative Frage. Die letzten signifikanten Beiträge gelangen Bilyk und Lacey für $d = 3$ (2008), sowie Bilyk, Lacey und Vagharshakyan für $d \geq 4$ (2008). Sie konnten die Existenz einer positiven Zahl η_d belegen, sodass für alle hinreichend großen N und allen N-Punktmengen

$$D_N^*(\mathcal{P}) \geq c_d \left(\log N\right)^{(d-1)/2+\eta_d}$$

gilt. In 2016 gelang es dem Autor dieser Dissertation eine Arbeit zu publizieren, in der – aufbauend auf ihren Methoden – verifiziert wird, dass diese Schranke für $d = 3$ mit $\eta_3 = 1/(32 + 4\sqrt{41}) - \epsilon \approx 0.017357\ldots$ und beliebigem $\epsilon > 0$ gilt. Um den Lesern einen klaren Überblick über die anspruchsvollen Methoden , welche von Bilyk und Lacey zum Teil neu und zum Teil wieder entdeckt wurden, sowie die Strategien, die zu dieser expliziten oberen Schranke führen, zu verschaffen, wird die diesbezügliche Arbeit des Autors hier gründlich und detailliert erörtert.

Im Hinblick auf den zweiten Hauptteil, nämlich der Konstruktion von gut verteilten Punktmengen und Folgen, verfolgt diese Dissertation den relativ modernen Zweig der so genannten hybriden Folgen. Grundsätzlich werden diese durch die Gegenüberstellung von Vertretern von (klassischen) niedrigdiskrepanten Folgen erzeugt. An ein gemeinsames Resultat von Hofer und dem Autor anknüpfend, welches bereits zur Publikation eingereicht wurde, wird hier die zweidimensionale Folge $(\boldsymbol{z}_k)_{k\geq 0}$, $\boldsymbol{z}_k = (x_k, \{k\alpha\})$, untersucht, wobei α eine irrationale Zahl in $(0,1)$ darstellt und $(x_k)_{k\geq 0}$ eine digitale Folge bezeichnet im Sinne von Niederreiter.

Die Konstruktion von $(x_k)_{k\geq 0}$ beruht per Definition auf einer unendlichen Matrix C mit Einträgen aus $\{0,1\}$. Zwei Spezialfälle solcher Folgen wurden bereits von Niederreiter (2009) und von Aistleitner, Hofer und Larcher (2016) untersucht. Wählt man einerseits für C die Einheitsmatrix (so genannte

Halton–Kronecker Folge), so verhält sich die Sterndiskrepanz von $(\boldsymbol{z}_k)_{k\geq 0}$ aus metrischer Sicht optimal. Nimmt man andererseits wieder die Einheitsmatrix an Stelle von C und ersetzt ihre erste Zeile durch eine unendliche Abfolge von Einsen (steht in Verbindung zur so genannten evil Kronecker Folge), so verschlechtert sich das metrische Verhalten von $(\boldsymbol{z}_k)_{k\geq 0}$ signifikant. Das zentrale Thema dieses Teils der Dissertation geht der Frage nach, was zwischen diesen beiden Fällen passiert. Genauer gesagt, werden solche Halton–Kronecker Folgen untersucht, bei denen die erste Zeile von C durch eine periodische Störung $(c_k)_{k\geq 0}$, welche aus Blöcken der Form $(1, 0, \ldots, 0)$ der Länge ℓ besteht, ersetzt wird. Das Resultat dieser Untersuchungen schlägt sich in scharfen Diskrepanzschranken für $(z_k)_{k\geq 0}$ im Falle, dass α beschränkte Kettenbruchkoeffizienten besitzt, nieder, die sich überraschender Weise mit abnehmender Dichte von Einsen in $(c_k)_{k\geq 0}$ verschlechtern. Außerdem erhält man starke metrische Schranken, welche das Verhalten von bereits bekannten Resultaten widerspiegeln, das heißt, die in den Schranken auftretenden Exponenten nähern sich dem (vermeintlich) optimalen Wert für $\ell \to \infty$.

Darüber hinaus weist dieses Thema eine enge Verbindung zu lakunären trigonometrischen Produkten der Form $\prod_{j=0}^{r-1} |\cos(2^j \alpha\pi + c_j \pi/2)|$ auf, weshalb auch hierfür scharfe allgemeine und starke metrische Schranken als kleiner Bonus in dem diesbezüglichen Kapitel bewiesen werden.

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- "Erzeugung normalverteilter Zufallszahlen und -vektoren", supervised by Gerhard Larcher, Johannes Kepler University Linz.
  English title: Construction of normally distributed random numbers and -vectors.

Publications

5. R. Hofer, F. Puchhammer: *Sharp general and metric bounds for the star discrepancy of perturbed Halton–Kronecker sequences.* Submitted. (Preprint available under `https://arxiv.org/abs/1611.01288`.)

4. F. Puchhammer: *On an explicit lower bound for the star discrepancy in three dimensions.* To appear in Mathematics and computers in simulation, Elsevier. (Preprint available under `http://arxiv.org/abs/1602.01307`.)

3. G. Larcher, F. Puchhammer: *An improved bound for the star discrepancy of sequences in the unit interval.* Uniform distribution theory 11 (2016), no. 1, 1–14.
(Preprint available under `http://arxiv.org/abs/1511.03869`.)

2. B. Eichinger, F. Puchhammer, P. Yuditskii: *Jacobi flow on SMP matrices and Killip-Simon problem on two disjoint intervals.* Comput. methods funct. theory 16 (2016), no. 1, 3–41.

1. N. I. Akhiezer: *A Generalization of a Minimal Problem of Korkin-Zolotarev kind.* arXiv:1401.5642v2 [math.CA], `http://arxiv.org/abs/1401.5642`, (2014)
(Translation from German into English of Akhiezer's historical paper "Verallgemeinerung einer Korkin-Zolotareffschen Minimum-Aufgabe".)

Workshop and conference attendances

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Contributed talk: *Sharp general and metric bounds for the star discrepancy of perturbed Halton–Kronecker sequences.*

- International conference on uniform distribution theory, Sopron, Hungary. (07/2016)

- Workshop on discrepancy theory, Varenna, Italy. (06/2016)
Invited talk: *Explicit lower bounds for the star discrepancy of sequences and point sets.*

- SFB Kooperationsworkshop, Linz, Austria. (04/2016)

- SFB Winter school on complexity and discrepancy, Traunkirchen, Austria. (12/2015)

- IMACS Seminar on Monte Carlo mehtods (MCM), Linz, Austria. (07/2015)

- Open problems workshop with Prof. Beck, Linz, Austria. (6/2015)

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- Workshop on the occasion of Harald Niederreiter's 70th birthday: Applications of algebra and number theory, Linz, Austria. (06/2014)
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- Tutorial: Analysis 1 (part of WS 2014).

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Acknowledgements

I would like to thank the Austrian research fund (FWF) for their financial support via the special research program (SFB) *Quasi-Monte Carlo Methods: Theory and Applications*, in particular the project F5507-N26, i.e. *Improved Discrepancy Estimates for Various Classes of Sequences*.

Secondly, I would like to express my sincere gratitude to my advisor Gerhard Larcher, not only for his continuous and substantial support of my studies, but also for his valuable advise on numerous occasions, especially with regard to my applications as a postdoc. Moreover, I truly value the freedom I was given to organize my work and the patience he showed everytime I had to navigate my way out of a dead end within my research, while providing help the second I needed it. I would say that I have learned a lot about the work as a mathematician and for me personally, simply because of this style of leadership.

That being said, I also have to thank Dmitriy Bilyk, Josef Dick, and Aicke Hinrichs for their support and help with my applications for postdoctoral positions.

Furthermore, I need to thank Roswitha Hofer for her efforts to profoundly introduce me to a comprehensive topic in such a short time and for her tremendous input to our joint work. Without her these results would still be beyond reach. Additionally, I want to thank Dmitriy Bilyk for taking the time to discuss and give further advise on my research during one of his visits in Linz. Special thanks also go out to Friedrich Pillichshammer for introducing me to discrepancy theory and quasi-Monte Carlo methods in the first place; as well as for advising me with my Bachelor's and Master's thesis and for pointing out this vacant PhD-position to me.

I also need to mention, that I have very much enjoyed working together with so many nice colleagues from my department, from the department of analysis and from the Ricam and I will certainly miss Altenbergerstraße's weekly luncheon.

Last but definitely not least I want to thank my family and friends for supporting me spiritually throughout my academic carreer. I especially want to mention my significant other, Kathrin Hiesl, and our two dogs Kenny and Flin. They helped me with everything they could. The success of this thesis is vastly due to Kathrin's unconditional support and the latter two did their best to keep me distracted.

Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe. Die vorliegende Dissertation ist mit dem elektronisch übermittelten Textdokument identisch.

Contents

Chapter 1

A brief guide to discrepancy theory

The motivation for studying discrepancy theory comes two-fold. As a first approach, it can be seen as an attempt to approximate continuous uniform distribution on the d-dimensional unit cube $[0,1)^d$ by deterministic point sets or sequences. The second approach comes from numerical mathematics and is described a couple of paragraphs further below. The first of these two incentives is followed within the next section of this thesis. The theory behind this approach is commonly referred to as *uniform distribution modulo one*, or shorter, *uniform distribution theory*. Usually, one would probably not go as far back as to uniform distribution theory in writing a paper on discrepancy theory. Since 2016 (the year I started to draft this work) was the centenniel of Hermann Weyl's celebrated paper *Über die Gleichverteilung von Zahlen mod. Eins* ([82]), which can be seen as the cradle of this field of mathematics, I decided to dedicate a small yet separate section to this topic.

In order to determine the quality of an approximation in the sense as described above, we consider the so-called discrepancy function.

Definition 1.1 (Discrepancy function, star discrepancy of point sets)**.** Let $\mathcal{P} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_N\}$ be an N-point set in $[0,1)^d$. For every $\boldsymbol{x} = (x_1, x_2, \dots, x_d) \in [0,1)^d$ we define the *discrepancy function* of $\mathcal{P}$ as

$$D_N(\mathcal{P}, \boldsymbol{x}) = \#\left(\mathcal{P} \cap [0, \boldsymbol{x})\right) - N\lambda_d\left([0, \boldsymbol{x})\right),$$

where we abbreviated $[0, \boldsymbol{x}) = [0, x_1) \times [0, x_2) \times \cdots \times [0, x_d)$ and where λ_d denotes the Lesbesgue measure on $\mathbb{R}^d$. Furthermore, we call its L^∞-norm the *star discrepancy* of $\mathcal{P}$, i.e.

$$D_N^*(\mathcal{P}) = \|D_N(\mathcal{P}, \cdot)\|_\infty = \sup_{\boldsymbol{x} \in [0,1)^d} |D_N(\mathcal{P}, \boldsymbol{x})|.$$

The minuend in the definition of the discrepancy function

$$\mathcal{A}(\mathcal{P}, N, \boldsymbol{x}) = \#\left(\mathcal{P} \cap [0, \boldsymbol{x})\right)$$

is commonly refered to as the *counting part*, whereas the subtrahend is usually called the *linear part*. Observe that the above definition admits a very pertinent interpretation. Let $I \subset [0,1)^d$ be an arbitrary axis-parallel box anchored at the origin. While the counting part gives the *actual* number of points that lie in I, the linear part indicates the *expected* number of points within I if we assume uniform distribution. Hence, the star discrepancy can be seen as a quality measure of our approximation, indeed.

In order to not obscure readers which are familiar to this field of mathematics it needs to be added that very often the discrepancy function (or at least the star discrepancy) is *normalized*, i.e. it is divided by N. Since the *unnormalized* version comes in more naturally and more conveniently within the approaches which are pursued here, we adhere to the definition above.

For sequences $\mathcal{S} = (s_k)_{k \geq 1}$ in $[0,1)^d$ this definition needs to be slightly altered. In this case, we define the N-th discrepancy function to be the discrepancy function of the truncated sequence. More precisely, we merely refine the definition of the counting part to

$$\mathcal{A}(\mathcal{S}, N, \boldsymbol{x}) = \#\left(\{s_k : 1 \leq k \leq N\} \cap [0, \boldsymbol{x})\right), \qquad \boldsymbol{x} \in [0,1)^d$$

and, subsequently, the discrepancy function (of the first N points) of $\mathcal{S}$ is given by

$$D_N(\mathcal{S}, \boldsymbol{x}) = \mathcal{A}(\mathcal{S}, N, \boldsymbol{x}) - N\lambda_d\left([0, \boldsymbol{x})\right), \qquad \boldsymbol{x} \in [0,1)^d.$$

Certainly, an analogous definition of the star discrepancy (or any other definition via a norm of the discrepancy function) applies.

It hardly comes as a suprise that the star discrepancy naturally occurs in uniform distribution modulo one and, consequently, discrepancy theory arises from this field of mathematics. This connection is outlined in the following Section 1.1. From another perspective, various norms of the discrepancy function (most prominently the star discrepancy) appear in error bounds for certain high-dimensional numerical integration algorithms, i.e. the so-called quasi-Monte Carlo (QMC) integration. This approach is briefly pursued in Section 1.2, thus concluding this motivational chapter.

Over time an extensive theory has evolved around the magnitude of D_N^* in terms of the number of points N. See, for instance, the books [18, 45, 54], just to name a few. This in turn raises two subcategories of questions. First of all, one tries to determine the exact growth rate for explicit point sets or sequences. In particular, finding a point set/sequence with the smallest possible star discrepancy is of special interest. This directly leads to the second type of questions, namely, what is the optimal rate of the star discrepancy? More precisely, this entails finding good lower bounds which hold for all point sets/sequences.

At this point it needs to be added that, due to Proinov, lower bounds for the discrepancy of arbitrary point sets in $[0,1)^d$, which are valid for sufficiently large N, automatically hold for arbitrary sequences in $[0,1)^{d-1}$ for infinitely many N. Conversely, due to a result by Roth [72] (see also [61, Lemma 3.7]), the existence of a $(d-1)$-dimensional sequence subject to a specific upper discrepancy bound implies the existence of a d-dimensional point set with the same estimate for its discrepancy. Therefore, the study of finite point sets and the study of infinite sequences are closely related.

The second of the aforementioned questions, i.e. finding optimal lower bounds, is unsolved for the most part and proves to be an intriguingly difficult task. Only for point sets in dimension $d = 2$ (and, hence, for sequences in the unit interval too) the exact rate of growth is known due to Schmidt 1972 (see [73]) and Halász 1981 (see [26]). The second chapter of this thesis emphasizes on tackling this long-standing open problem. After setting out the most celebrated and relevant historic and current developments towards this direction, the subsequent sections focus on two new contributions by the author, one for sequences in $d = 1$ ([48]) and another one for point sets in $d = 3$ ([69]).

With a view to the first question posed above, i.e. determining the exact growth of D_N^* for specific point sets or sequences, one can rely on numerous studies of various *good* examples, ranging from Kronecker sequences, over to Halton sequences and Hammersley point sets, to digital sequences and (t, m, d)-nets, and many more. Perhaps the most prominent classical examples are dealt with in [18, 45]. The first type of sequences from this list is introduced in Section 1.1 already, as it constitutes the very origin of uniform distribution theory. In the final chapter we augment the list of explicit examples by digital sequences and by Halton sequences as a special instance of these. Furthermore, we give detailed information on the distribution properties of these sequences in Section 3.1. Following an idea arising from the approach via QMC integration, which was first formulated by J. Spanier in [78], we thus combine Kronecker sequences with certain digital sequences to form a new class of (so-called *hybrid*) sequences and present their concise distributional behavior in Section 3.2 (see [37]).

1.1 A first approach: Uniform distribution modulo one

As it was already mentioned in the introduction, we are interested in finding sequences which are *evenly* or, in other words, *uniformly* distributed in the d-dimensional unit cube. For an extensive survey the reader is once again encouraged to study the monograph [45], where more details and further information on the topics, which are presented here in all brevity, can be found. In more mathematical terms the desired quality of being *uniformly distributed* is described as follows.

Definition 1.2 (Uniform distribution modulo 1)**.** A sequence $\mathcal{S} = (s_k)_{k\geq 1}$ is said to be *uniformly distributed modulo 1* (u.d. mod 1) iff for all axis parallel boxes $J \subseteq [0,1)^d$ we have

$$\lim_{N\to\infty} \frac{\mathcal{A}(\mathcal{S}, N, J)}{N} = \lambda_d(J), \tag{1.1}$$

where $\mathcal{A}(\mathcal{S}, N, J)$ denotes the obvious extension of the counting part to boxes, i.e.

$$\mathcal{A}(\mathcal{S}, N, J) = \#\left(\{s_k : 1 \leq k \leq N\} \cap J\right).$$

Observe that for a sequence to be u.d. mod 1 is an even stronger property than to be merely dense. Indeed, it can be shown that every u.d. sequence is dense in $[0,1)^d$. Whereas the sequence $(\{\sin(k)\})_{k\geq 1}$, with $\{\theta\}$ denoting the fractional part of $\theta \in \mathbb{R}$, is dense in $[0,1)$ but certainly not u.d. mod1 (cp. [45, Ch.1, Exercise 2.7]).

From a historical perspective, in the beginning, the main object of interest was the so-called *Kronecker sequence*, which we define below.

Definition 1.3. Let $\boldsymbol{\alpha} \in [0,1)^d$. Then the d-dimensional Kronecker sequence is defined by $(\{\boldsymbol{\alpha} k\})_{k\geq 0}$, where the fractional part is taken componentwise.

In one dimension it is evident that the Kronecker sequence runs through the same orbit over and over again if α is rational, hence we may confine ourselves to irrational α's, as this is the only interesting case. In arbitrary dimensions d the corresponding condition is that the numbers $1, \alpha_1, \alpha_2, \ldots, \alpha_d$ are linearly independent over the rationals $\mathbb{Q}$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d)$.

The central question is, are these conditions also sufficient for the Kronecker sequence to be u.d. mod1? Although a positive answer was already given by Bohl in 1909, it was Weyl who took up this question in 1916 and built a sound theory around it. In the end, the proof of this fundamental result requires (at most) one line and the entire theory behind it already hinted various insights to other fields of mathematics, which were completely unknown at that time.

Let us continue our survey by noticing that (1.1) admits of the equivalent formulation

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_J(s_k) = \int_{[0,1)^d} \mathbb{1}_J(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x},$$

where $\mathbb{1}_J$ denotes the indicator function of J. Having established this relation one might already guess that the next step would be to extend this result from *step functions* to Riemann integrable functions. Indeed, it can be shown that a sequence $(s_k)_{k\geq 1}$ in $[0,1)^d$ is uniformly distributed if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(s_k) = \int_{[0,1)^d} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \tag{1.2}$$

for all Riemann integrable functions $f : [0,1]^d \to \mathbb{R}$, see [45, Ch.1, Theorem 1.1], for instance. Observe that, assuming the Kronecker sequence is u.d. mod1 for some $\boldsymbol{\alpha} \in \mathbb{R}^d$ (which we will see a few lines further below), this may already be seen as a precursor of Birkhoff's ergodic theorem, as Aistleitner mentioned in a talk at Stanford on Weyl and his 1916 paper in 2016. Moreover, the above relation already hints that u.d. sequences can be employed successfully as integration nodes in numerical algorithms and thus gives birth to the idea of QMC methods. It also needs to be mentioned that it is impossible to extend this characterization to Lesbesgue integrable functions.

In order to determine whether an explicit sequence is u.d. mod1, however, the above criterion is still very impractical. Therefore, another reformulation is required. More specifically, it can be shown that the sequence $(s_k)_{k\geq 1}$ is u.d. mod1 if and only if (1.2) holds for all continuous functions $f : [0,1]^d \to \mathbb{C}$ with period one. Following the quote from his paper "Die einfachste Funktion von der Periode 1 ist $\mathrm{e}^{2\pi \mathrm{i} x}$." which translates to "The simplest function of period one is $\mathrm{e}^{2\pi \mathrm{i} x}$." Weyl discovered an extremely strong and convenient characterization of u.d. sequences, the so-called *Weyl criterion*, via a trigonometric version of Weierstrass' approximation theorem.

Theorem 1.4 (Weyl criterion, cf. [82], [45, Ch.1, Theorem 2.1])**.** *A sequence* $(s_k)_{k\geq 1}$ *in* $[0,1)^d$ *is u.d.* mod1 *if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h}\cdot s_k} = 0$$

for all $\boldsymbol{h} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$.

It is now an almost trivial task to finish the study of our model problem: the distribution of the Kronecker sequence.

Corollary 1.5 (Cf. [45, Ch.1, Example 6.1])**.**
The Kronecker sequence $(\{\boldsymbol{\alpha} k\})_{k\geq 0}$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d$, is u.d. mod 1 if and only if the numbers $1, \alpha_1, \alpha_2, \ldots, \alpha_d$ are linearly independent over $\mathbb{Q}$.

Proof. The "only if" part has already been dealt with in the paragraph following Definition 1.2. Assuming that $1, \alpha_1, \alpha_2, \ldots, \alpha_d$ are linearly independent over the rationals it is obvious that $\boldsymbol{h} \cdot \boldsymbol{\alpha} \notin \mathbb{Q}$ for all $\boldsymbol{h} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$. Consequently,

$$\left|\frac{1}{N}\sum_{k=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i}\boldsymbol{h}\cdot\{\boldsymbol{\alpha} k\}}\right| = \left|\frac{1}{N}\sum_{k=0}^{N-1} \mathrm{e}^{k 2\pi \mathrm{i}\boldsymbol{h}\cdot\boldsymbol{\alpha}}\right| = \left|\frac{\mathrm{e}^{N 2\pi \mathrm{i}\boldsymbol{h}\cdot\boldsymbol{\alpha}} - 1}{N(\mathrm{e}^{2\pi \mathrm{i}\boldsymbol{h}\cdot\boldsymbol{\alpha}} - 1)}\right|$$
$$\leq \frac{2}{N(\mathrm{e}^{2\pi \mathrm{i}\boldsymbol{h}\cdot\boldsymbol{\alpha}} - 1)} \xrightarrow[N\to\infty]{} 0.$$

□

Let us now draw the connection to discrepancy theory. It can be shown (see [45, Ch.2, Theorem 1.1], for instance) that we have

$$\mathcal{S} \text{ is u.d. mod } 1 \qquad \Longleftrightarrow \qquad \lim_{N\to\infty} \frac{D_N^*(\mathcal{S})}{N} = 0,$$

for every sequence $\mathcal{S}$ in $[0,1)^d$. I.e., the star discrepancy grows more slowly than N for u.d. sequences . This directly leads to the question of how uniformly a sequence can actually be distributed or what is the speed of convergence for explicit sequences? Maybe there are even *perfect* sequences for which the approximation error in the sense of D_N^* remains bounded. A negative (yet not final) answer to the latter question will be given in the next chapter and constitutes one of the central principles of *irregularities of distribution*. The quest for discrepancy bounds for explicit u.d. sequences will be accommodated in Chapter 3, where we encounter the model problem – the Kronecker sequence – again, along with other fascinating related examples.

1.2 A second approach: Quasi-Monte Carlo methods

As it has already been mentioned several times and implicitely hinted in (1.2) a second motivation for discrepancy theory comes from numerical integration.

For a comprehensive survey the reader is refered to [18, 61]. Here, the goal is to approximate the integral of functions $f : [0,1]^d \to \mathbb{R}$ belonging to a *reasonable* class of functions F by an equally weighted quadrature rule, i.e.

$$I_d(f) = \int_{[0,1)^d} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \approx \frac{1}{N}\sum_{p\in\mathcal{P}} f(p) =: Q_{N,\mathcal{P}}(f), \tag{1.3}$$

where $\mathcal{P} \subseteq [0,1)^d$, $\#\mathcal{P} = N$, denotes a multiset of N integration nodes. Certainly, the integration error

$$R_{N,\mathcal{P}}(f) = |I_d(f) - Q_{N,\mathcal{P}}(f)|$$

should depend on the function f and the nodes $\mathcal{P}$. Within this approach, we try to choose $\mathcal{P}$ suitably for a given function space F. The famous introductory example of taking $F = L^2$, for instance, and $p_1, p_2, \ldots, p_d$ uniformly and independently distributed in $[0,1)^d$ was already considered in [18, 61]. By rather elementary computations they obtain

$$\mathbb{E}\left(R_{N,\mathcal{P}}(f)\right) \le \sqrt{\mathbb{E}\left(R^2_{N,\mathcal{P}}(f)\right)} \le \frac{\sigma(f)}{\sqrt{N}},$$

where $\mathbb{E}(\cdot)$ denotes the mean and where the standard deviation σ is defined as

$$\sigma(f) = \sqrt{\mathbb{E}\left(f - \mathbb{E}(f)\right)^2}.$$

Observe that, quite surprisingly, the expected error is independent of the dimension. The approach of taking uniformly distributed independent sample points is commonly referred to as *Monte Carlo* integration.

In an attempt to accelerate the speed of convergence or to meet the requirements of other function spaces more flexibly one might try to choose the integration nodes deterministically. This strategy is the distinctive feature of QMC methods. In this case we may state the celebrated results by Koksma ([41]) for $d = 1$ and Hlawka ([31]) for $d \ge 2$.

Theorem 1.6 (Koksma–Hlawka inequality)**.** *Let $f : [0,1]^d \to \mathbb{R}$ be a function of bounded variation in the sense of Hardy and Krause. Then*

$$R_{N,\mathcal{P}}(f) \le c_f \frac{D^*_N(\mathcal{P})}{N},$$

where $c_f > 0$ is independent of $\mathcal{P}$.

Hence, the star discrepancy of the point set under consideration directly determines the quality of the integration algorithm. At this point we preempt results from Chapter 3, stating that there exist point sets $\mathcal{P}_0$ satisfying

$D_N^*(\mathcal{P}_0) \leq c_d(\log N)^{d-1}$, $c_d > 0$ independent of N, so-called *low-discrepancy point sets*. In total this yields an upper bound for the integration error of $c_{f,d}(\log N)^{d-1}/N$, $c_{f,d} > 0$ independent of N.

We immediately see that, asymptotically, the convergence rate is much better than in the pure Monte Carlo setting, however, the dimension enters critically for applications. This means that, in order to exploit the faster rate one has to employ exponentially (in d) many points. This phenomenon is usually called the *curse of dimensionality* and is the center of studies in the field of *information based complexity* in a more general setting. For more information on this topic the reader is referred to the well-known trilogy by Novak and Woźniakowski [64–66].

It needs to be added that the same theory can be established for truncated infinite sequences such as the first N points of the Kronecker sequence, for instance. This, of course, leads to a worse discrepancy behavior, i.e. we need to replace d by $d+1$ in the upper bounds. The upside to this approach, however, is that we find ourselves in a *dynamic* setting. Indeed, if we want to increase the number of integration nodes one simply computes the additional elements of the sequence and appends them to the old ones. For point sets the situation is quite different. Here, usually *all* the nodes change, so the entire point set has to be recalculated.

As a summary one might say that deterministic point sets or sequences, which are *evenly spread* across the d-dimensional unit cube, are well suited to be employed in numerical integration algorithms of the form (1.3). The quality of these algorithms is in turn bounded by the equidistribution qualities of the point set, in other words, by its star discrepancy. Therefore it is of major interest to pick low discrepancy point sets or sequences.

Chapter 2

Lower bounds for the star discrepancy

It is one of the basic principles of *irregularities of distribution* that no N-point set (or sequence) can be distributed too well, i.e., its star discrepancy increases with N. The name of this principle reaches back to Roth who also provided one of the most famous results that support this observation in his seminal paper [72] (1954), see Theorem 2.1 below. Finding the exact optimal growth rate, however, appears to be extraordinarily difficult and has left many mathematicians pondering for more than 60 years now and will probably continue to do so in the near future.

The reasons that account for the challenging nature of the problem can be set out very clearly. Namely, for any N-point set $\mathcal{P}$ one has to prove the existence of an axis parallel box anchored at the origin, for which the corresponding discrepancy function is large. Moreover, this has to be done without having any structual information on $\mathcal{P}$. On the other hand, this largely contributes to the beauty of this field, since innovative techniques and the interplay of methods from across the board of disciplines, ranging from functional and harmonic analysis to number theory and combinatorics, appear to be essential for obtaining good results.

Non surprisingly, the absence of sharp results has been the breeding ground for speculations and conjectures. Within the following Section 2.1 we shed light on the historic develoments in the field of irregularities of distribution, not only from a quantitative perspective, but also touch upon the evolution of techniques. In the course of this historical outline also the best known conjectures are stated.

Section 2.2 presents a result of the author together with his supervisor Larcher for sequences in the unit interval, following an ingenious approach of Liardet ([50]) who unfortunately deceased in 2014. Hence, the underlying paper [48] was published in a special issue of *Uniform Distribution Theory* dedicated to his memory.

One of the major contributions to irregularities of distribution was started by Roth in 1954 and is characterized by applying tools from harmonic analysis. In Section 2.3 we describe specific features of this approach by demonstrating Halász' proof of Theorem 2.2 and present its (everything but trivial) extension to point sets in three dimensions (Theorem 2.5). This in turn is done by giving a concise proof of the author's main contribution to this topic, i.e. of Theorem 2.23, which is accepted for publication in *Mathematics and Computers in Simulation*, see [69].

2.1 Irregularities of distribution – A historical outline

Let us begin our survey with Roth's seminal paper [72] from 1954, in which he proved the following theorem.

Theorem 2.1 (Roth, 1954). *For all $d \geq 2$ and all N sufficiently large, any N-point set $\mathcal{P}$ satisfies*

$$\|D_N(\mathcal{P}, \cdot)\|_2 \geq c_d \, (\log N)^{\frac{d-1}{2}} ,$$

with $c_d > 0$ independent of N.

It needs to be added that the L^2-norm of the discrepancy function is bounded from above by the L^∞-norm, i.e., the star discrepancy of $\mathcal{P}$. Considering the facts that the L^2-norm behaves far more averaging than the L^∞-norm and that the discrepancy function can be highly localized, it hardly comes as a surprise that this bound is now known not to be sharp (see Schmidt's Theorem 2.2 below). Yet, it is sharp for the L^2-discrepancy, as can be seen by taking a shifted version of the Hammersley point set, cf. [54, Theorem 2.5] or [30] for a more recent proof, which considers L^p-norms for $1 < p < \infty$ in $d = 2$. Nevertheless, it was Roth's approach which struck a chord at that time. His idea was to incorporate tools from harmonic analysis to discrepancy theory via the Haar function system. This lead to a completely new methodology for proving discrepancy bounds. The tremendous impact of Roth's idea is very well captured in the survey paper [8], which, by the way, also contains a proof of Theorem 2.1.

It took as much as 18 years until a better estimate for D_N^* for one dimensional sequences was discovered by Schmidt, see [73].

Theorem 2.2 (Schmidt, 1972). *For every sequence in the unit interval we have*

$$D_N^*(\mathcal{S}) \geq c \log N$$

for infinitely many N with an absolute constant $c > 0$.

Now, this bound is even known to be sharp. Indeed, examples of sequences which satisfy $D_N^*(\mathcal{S}) \leq C \log N$, $C > 0$, go back even further to Lerch [49] (1904) or Van der Corput [80] (1935). Later, in 1981, Halász managed to give a proof of Schmidt's result (as a matter of fact, he considered point sets in $d = 2$) by refining Roth's approach via introducing special auxiliary functions, namely Riesz products, and using duality, see [26]. We will return to the proof of Halász in Section 2.3, as it is essential to understand the basic ideas behind this result in order to grasp the concept of Bilyk's and Lacey's proof of Theorem 2.5 and its quantification Theorem 2.23.

Schmidt's result in combination with the fact that no explicit d-dimensional sequence or point set could be found (and still has not been found) for which the exponent of the logarithmin its discrepancy bound is smaller than d or $d-1$, respectively, lead to the perhaps most widely accepted conjecture below.

Conjecture 2.3. *For all dimensions d any d-dimensional point set $\mathcal{P}$ or sequence $\mathcal{S}$ is subject to*

$$D_N^*(\mathcal{P}) \geq c_d (\log N)^{d-1} \qquad \text{or} \qquad D_N^*(\mathcal{S}) \geq c_d (\log N)^d$$

for all N sufficiently large or for infinitely many N, respectively, for some constant $c_d > 0$ independent of N.

Unfortunately, Halász' methods are not directly applicable to higher dimensions due to a shortfall of certain orthogonality properties and, for that matter, of Hilbert space specific features. This shortfall was first successfully tackled for $d = 3$ by Beck in [4]. By combining Halász' approach with graph theory and altering the auxiliary function pertinently he managed to give the first improvement to Roth's bound in dimension three in 33 years by a double-logarithmic factor.

Theorem 2.4 (Beck, 1989). *For all N-point sets $\mathcal{P}$ and all $\epsilon > 0$ we have*

$$D_N^*(\mathcal{P}) \geq c_\epsilon \log N \cdot (\log \log N)^{\frac{1}{8} - \epsilon}$$

for N sufficiently large and where $c_\epsilon > 0$ is independent of N.

It took another 19 years until a major improvement to this result as well as to Roth's theorem for $d \geq 4$ emerged. Without going into too many details at this point (we deal with them in Section 2.3), it were Bilyk and Lacey who refined Halász' auxiliary function in a way to handle the aforementioned shortfalls better and incorporated alternative tools (Littlewood–Paley inequalities) to make this handling more efficient. In [11] ($d = 3$) and together with Vagharshakyan in [13] ($d \geq 4$) they showed the following theorem below.

Theorem 2.5 (Bilyk, Lacey, Vagharshakyan, 2008)**.** *For $d \geq 3$ there exists a constant $\eta_d > 0$ such that any point set $\mathcal{P} \subseteq [0,1)^d$ consisting of N points, with N sufficiently large, satisfies*

$$D_N^*(\mathcal{P}) \geq c_d \left(\log N\right)^{\frac{d-1}{2}+\eta_d}, \qquad c_d > 0.$$

As a matter of fact, the papers [11, 13] are concerened with a different topic, namely the so-called *small ball inequality* (SBI). However, it appears that the proof techniques are almost identical and, hence, the discrepancy estimates more or less emerge as a side perk. The SBI itself emphasizes on lower bounds for the L^∞-norm of sums of the form

$$\sum_{|R|=2^{-n}} \alpha(R) h_R,$$

where the sum runs over all dyadic rectangles R in $[0,1)^d$ (with fixed volume 2^{-n}) and where h_R denotes the multivariate Haar function associated to R and $\alpha(R)$ some real coefficient. For further explanations of these terms see Definition 2.24. In two dimensions there is a rigorous proof for a special instance of the SBI, i.e. the signed SBI or SSBI, where one additionally assumes $|\alpha(R)| = 1$. Indeed, in [9] Bilyk and Feldheim show that in $d = 2$ the sets which minimize the above sums in the signed setting are in one-to-one relation to so-called digital $(0, n+1, 2)$-nets, i.e. a finite analogon to the infinite sequences considered in Section 3.1.2. Unfortunately, for $d \geq 3$ this connection remains purely heuristic. Nevertheless, these heuristics give rise to yet another famous conjecture for the star discrepancy, which is due to corresponding conjectures for the SSBI. As a consequence of the similarities in the methodology this conjecture should at least indicate the limitations of Roth's and Halász' orthogonal functions method.

Conjecture 2.6. *For all dimensions d any d-dimensional point set $\mathcal{P}$ or sequence $\mathcal{S}$ is subject to*

$$D_N^*(\mathcal{P}) \geq c_d \left(\log N\right)^{\frac{d}{2}} \qquad \text{or} \qquad D_N^*(\mathcal{S}) \geq c_d \left(\log N\right)^{\frac{d+1}{2}}$$

for all N sufficiently large or infinitely many N, respectively, with some constant $c_d > 0$ independent of N.

For the sake of completeness we state one more conjecture which, according to [8], is proposed by Skriganov.

Conjecture 2.7. *For all dimensions d any d-dimensional point set $\mathcal{P}$ or sequence $\mathcal{S}$ is subject to*

$$D_N^*(\mathcal{P}) \geq c_d \, (\log N)^{\frac{d-1}{2}+\frac{d-1}{d}} \qquad \text{or} \qquad D_N^*(\mathcal{S}) \geq c_d \, (\log N)^{\frac{d}{2}+\frac{d}{d+1}}$$

for all N sufficiently large or infinitely many N, respectively, with some constant $c_d > 0$ independent of N.

Observe that for the solved case, i.e. two-dimensional point sets, all of the conjectures coincide.

2.2 Sequences in the unit interval

In the previous section it is pointed out that the problem of determining the right order of growth of the star discrepancy of sequences in the unit interval has already been settled. Hence, one might even go one step further and ask for the best possible constant c^* satisfying the bound in Schmidt's Theorem 2.2. This gives rise to the following definition.

Definition 2.8 (Star discrepancy constant)**.** We call the number

$$c^* = \inf_{\mathcal{S}} \limsup_{N \to \infty} \frac{D_N^*(\mathcal{S})}{\log N},$$

where the infimum is taken over all sequences $\mathcal{S}$ in $[0,1)$, the *one-dimensional star discrepancy constant.*

To the author's best knowledge the record for the upper bound is currently held by Ostromoukhov (see [67]) who proved

$$c^* \leq 0.222223\ldots,$$

thereby improving earlier results by Faure from 1992 (see [21]).

A long standing best lower bound was established by Béjian in 1979 amounting to approximately 0.060... (see [6]) until Larcher followed a technique originally introduced by Liardet ([50]) and also used by Tijdeman and

Wagner ([79]) and, first of all, gave an illustrative and considerably simpler proof of Béjian's result and, secondly, improved the bound for the star discrepancy constant to

$$c^* \geq 0.0646363\ldots, \tag{2.1}$$

see [46].

Following Larcher's approach, he together with the author slightly improved upon this bound in [48]. The corresponding result is stated in the theorem below.

Theorem 2.9 (Cf. [48, Theorem 1]). *The one-dimensional star discrepancy constant satisfies the lower bound*

$$c^* \geq 0.065664679\ldots$$

The main content of this section is dedicated to giving a proof of this result, closely following the methods from [46] and [48]. To this end, we first of all set out all preliminaries in the following Section 2.2.1. Subsequently, we transfer the problem of finding the constant from the claim to a minimization problem over the space of so-called *admissible functions*, which will be introduced in Section 2.2.2. Then we solve this minimization problem or, to be more precise, we estimate its solution in Section 2.2.3 and thus conclude the proof of Theorem 2.9. Finally, Section 2.2.4 gives an example on how such a minimizer might look like and discusses related open questions.

2.2.1 Preliminaries and proof of Theorem 2.9

We consider an arbitrary sequence $\mathcal{S}$ in $[0,1)$. Furthermore, we fix $N = \lfloor a^t \rfloor$ for some real number a, $3 \leq a \leq 3.7$, and some positive integer t. Additionally, we divide the index set $A = \{1, 2, \ldots, N\} =: [\![N]\!]$ into subsets A_0, A_1, and A_2 in the following way.

$$A_0 = [\![\lfloor a^{t-1} \rfloor]\!], \quad A_2 = \{\lfloor a^t \rfloor - \lfloor a^{t-1} \rfloor + 1, \lfloor a^t \rfloor - \lfloor a^{t-1} \rfloor + 2, \ldots, \lfloor a^t \rfloor\},$$
$$A_1 = A \setminus (A_0 \cup A_2),$$

i.e., A_0 contains the first $\lfloor a^{t-1} \rfloor$ indices, A_2 the last $\lfloor a^{t-1} \rfloor$ indices and A_1 contains everything that lies in between.

For simplicity let us assume that $N = a^t$. As the difference to the actual value of N is of negligible size for large t, we compensate for this simplification by introducing an arbitrarily small correcting parameter $\epsilon > 0$ at the relevant stage of the proof.

We define the function $P(t)$ by

$$P(t) = \int_0^1 \left(\max_{i \in A} D_i(\mathcal{S}, x) - \min_{i \in A} D_i(\mathcal{S}, x) \right) \mathrm{d}x$$

and estimate

$$\begin{aligned} 2P(t) \geq & \int_0^1 \left(\max_{i \in A_2} D_i(\mathcal{S}, x) - \min_{i \in A_2} D_i(\mathcal{S}, x) \right) \mathrm{d}x \\ & + \int_0^1 \left(\max_{i \in A_0} D_i(\mathcal{S}, x) - \min_{i \in A_0} D_i(\mathcal{S}, x) \right) \mathrm{d}x \\ & + \int_0^1 \left| \max_{i \in A_2} D_i(\mathcal{S}, x) - \max_{i \in A_0} D_i(\mathcal{S}, x) \right| \mathrm{d}x \\ & + \int_0^1 \left| \min_{i \in A_2} D_i(\mathcal{S}, x) - \min_{i \in A_0} D_i(\mathcal{S}, x) \right| \mathrm{d}x, \end{aligned} \tag{2.2}$$

cf. [46, Lemma 2.1]. For a moment, let us assume that each of the last two summands is uniformly bounded in t from below by a certain constant, say, $b(a)$. We establish these facts later in Lemma 2.21 together with Lemma 2.15. Observe that, since $\#A_0 = \#A_2 = a^{t-1}$, the remaining two first summands both resemble the function P evaluated at $t-1$. Hence, by our assumption, we inductively obtain

$$P(t) \geq P(t-1) + b(a) \geq \ldots \geq tb(a) = \frac{\log N}{\log a} b(a).$$

On the other hand, quite obviously we have

$$P(t) \leq 2 \left| D_\nu(\mathcal{S}, \xi) \right|$$

for some $\xi \in [0,1]$ and some $\nu \leq N$. After taking the supremum w.r.t. ξ we may summarize as follows. For all N there exists a $\nu \leq N$ such that

$$D_\nu^*(\mathcal{S}) \geq \frac{P(t)}{2} \geq \log N \frac{b(a)}{2 \log a} \geq \log \nu \frac{b(a)}{2 \log a}.$$

Moreover, considering increasing N, it is evident that there are infinitely many ν satisfying this inequality. Noticing that the fourth summand in (2.2) can be treated similarly to the third summand we have thus shown the following crucial lemma.

Lemma 2.10 (Cf. [46, Proof of Theorem 1.1])**.** *Let*

$$f(x) = \max_{i \in A_2} D_i(\mathcal{S}, x) - \max_{i \in A_0} D_i(\mathcal{S}, x).$$

Under the assumption that $\|f\|_1 \geq b(a)$ *we have*

$$c^* \geq \frac{b(a)}{2\log a}.$$

The proof of Theorem 2.9 can now be easily derived.

Proof of Theorem2.9. As a result of Lemma 2.10 and Lemma 2.21 we have

$$\begin{aligned}\int_0^1 f(x)\,\mathrm{d}x &\geq t\|f^*\|_1 \geq \frac{\log N}{2\log a} b(a)\\ &= \frac{\log N}{2\log a}\left(\frac{(a-2)\left(12a+9+(a-2)(4a-3)\log\left(1+\frac{1}{a-2}\right)\right)}{a\left(a-\frac{1}{2}\right)^2\left(3+(a-2)\log\left(1+\frac{1}{a-2}\right)\right)} - \epsilon\right)\end{aligned}$$

for all $3 < a \leq 3.7$ and t sufficiently large, where f^* is defined in (2.5). Choosing $a = 3.62079\ldots$ in the latter expression yields the claimed bound for c^*. □

2.2.2 The space of admissible functions and an L^1-minimization problem

Besides Lemma 2.10, the second main ingredient to the proof is to transfer the problem of finding a uniform lower bound for $\|f\|_1$ to a minimization problem over a certain function space $\mathscr{F}$. This function space itself is defined by properties of f. The idea is to describe f (and thus $\mathscr{F}$) as precisely as possible, however, still ensuring that $\mathscr{F}$ is closed in a suitable topology.

Let us begin by gathering information on f. To this end let $i_0 = i_0(x) \in A_0$ and $i_2 = i_2(x) \in A_2$ such that $D_{i_0}(x, \mathcal{S}) = \max_{i\in A_0} D_i(x, \mathcal{S})$ and $D_{i_2}(x, \mathcal{S}) = \max_{i\in A_2} D_i(x, \mathcal{S})$, respectively. We may thus rewrite f as

$$f(x) = \mathcal{A}(\mathcal{S}, i_2, x) - \mathcal{A}(\mathcal{S}, i_0, x) - x(i_2 - i_0).$$

Obviously, $f(0) = f(1) = 0$. Furthermore, we immediately see that

$$a^{t-1}(a-2) = (a^t - a^{t-1}) \leq i_2 - i_0 \leq a^t,$$

hence, the slope of f is bounded and negative. Also, f itself is bounded in modulus by a^t.

It is an immediate observation that f is continuous between every two elements of our truncated sequence $\mathcal{S}_N = (s_i)_{1\leq i\leq N}$, even though $i_0(x)$ and $i_2(x)$ may change their values. Outside the points of $\mathcal{S}_N$ we can even say that f is linear. At all $x \in \{s_1, \ldots, s_N\}$, however, f might experience a jump.

Therefore, the number of discontinuities is at most $N = a^t$. Nevertheless, we always have $\lim_{x\uparrow s_i} f(x) = f(s_i)$, $1 \le i \le N$.

Let us now assume $i \in A_1$. Obviously, $\mathcal{A}(\mathcal{S}, i_0, \cdot)$ does not change its value when passing through s_i as $i > i_0$, while $\mathcal{A}(\mathcal{S}, i_2, \cdot)$ increases by 1. Hence, f has at least $\#A_1 = a^{t-1}(a-2)$ jumps of hight at least 1.

These properties would suffice to derive Béjian's bound, as it was shown in [46]. Here, we enhance our list by two additional, less obvious, characteristics of f to obtain a better result. The first enhancement alone yields (2.1), cf. [46, Remark 1], and together with the second one we arrive at the bound from Theorem 2.9, cf. [48, Lemma 1].

Lemma 2.11. *Let $[\alpha, \beta] \subseteq [0,1]$ such that f has a jump in $s_i \in (\alpha, \beta)$, $1 \le i \le N$, and no further elements of $\mathcal{S}_N$ lie in $[\alpha, \beta]$. Furthermore, let $x_1, x_2 \in [\alpha, s_i)$. Then the slope of f at x_1 can differ at most by a^{t-1} from the slope of f at x_2. Naturally, the same holds for $x_1, x_2 \in (s_i, \beta]$.*

Proof. Let $x \in (\alpha, s_i)$ and $j \in \{0, 2\}$. The claim is trivial if i_j is constant on $[\alpha, s_i)$. Hence, we assume that i_j changes its value in x. W.l.o.g. let $x_1 < x < x_2$. This implies that

$$D_{i_j(x_2)}(\mathcal{S}, x_2) > D_{i_j(x_1)}(\mathcal{S}, x_2) \qquad \text{and} \qquad D_{i_j(x_2)}(\mathcal{S}, x) = D_{i_j(x_1)}(\mathcal{S}, x).$$

Since x is not an element of the truncated sequence $\mathcal{S}_N$, the counting parts remain constant traversing through x, i.e.

$$\mathcal{A}(\mathcal{S}, i_j(x_2), x) = \mathcal{A}(\mathcal{S}, i_j(x_2), x_2) \qquad \text{and} \qquad \mathcal{A}(\mathcal{S}, i_j(x_1), x) = \mathcal{A}(\mathcal{S}, i_j(x_1), x_2).$$

As a combination of the above two (in)equalities we thus obtain

$$x_2(i_j(x_1) - i_j(x_2)) > x(i_j(x_1) - i_j(x_2)).$$

Consequently, $i_j(x_1) - i_j(x_2) > 0$, i.e. $i_j(\cdot)$ is decreasing, and hence

$$\begin{aligned} &|(i_2(x_2) - i_0(x_2)) - (i_2(x_1) - i_0(x_1))| \\ &\qquad \le \max\{i_2(x_1) - i_2(x_2), i_1(x_1) - i_1(x_2)\} \le a^{t-1}. \end{aligned}$$

□

Summarizing, we have gathered information on jumps of f resulting from point s_i, $i \in A_1$ and on kinks. We now present the aforementioned second additional property of f which tackles those discontinuities of f which are due to elements of the sequence indexed by A_2.

Lemma 2.12. *Let $i \in A_2$, i.e., $i = a^t - a^{t-1} + k$ for some integer k, $1 \leq k < a^{t-1}$, and assume that f has a discontinuity in s_i, i.e. the i-th element of our truncated sequence $\mathcal{S}_N$. Let further l_i, $r_i \in A$ such that $\mathcal{S}_N \cap (s_{l_i}, s_{r_i}) = \{s_i\}$. If there exists an $\overline{x} \in (x_i, x_{r_i})$ such that, f has slope $\sigma(\overline{x}) > \sigma_0 - k$ in $\overline{x}$, then*

$$f(\underline{x}) \geq f(\overline{x}) - \sigma_0(\overline{x} - \underline{x}), \qquad \forall \underline{x} \in [s_{l_i}, s_i).$$

Here, $\sigma_0 = -a^{t-1}(a-2)$ denotes the largest possible slope.

Remark 2.13. The meaning of Lemma 2.12 is illustrated in Figure 2.1. Using the same notation as above, $f(\underline{x})$ lies above the line with slope σ_0 reaching back from the point $(\overline{x}, f(\overline{x}))$ (dashed) in case the slope of f (solid) becomes flatter than $\sigma_0 - k$.

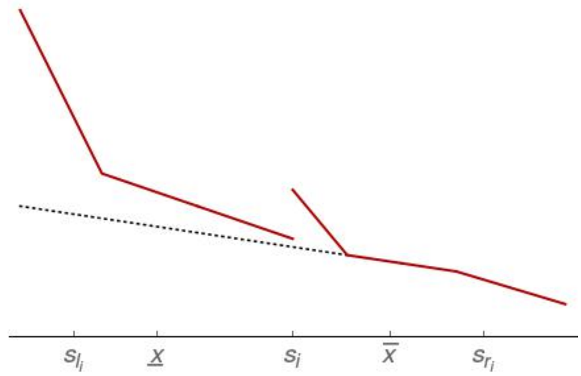

Figure 2.1: Illustration of Lemma 2.12.

Proof of Lemma 2.12. Let $\underline{x}$, $\overline{x}$ be like above with $\sigma(\overline{x}) > \sigma_0 - k$. First we show $i_2(\overline{x}) < i$. Indeed, we have

$$a^{t-1} - i_2(\overline{x}) \geq i_0(\overline{x}) - i_2(\overline{x}) = \sigma(\overline{x}) > \sigma_0 - k = -a^{t-1}(a-2) - k$$

and, hence, $i_2(\overline{x}) < a^t - a^{t-1} + k = i$. This in turn implies that $\mathcal{A}(\mathcal{S}, i_2(\overline{x}), \cdot)$ does not change its value in x_i or, in other words, $D_{i_2(\overline{x})}(\mathcal{S}, \cdot)$ does not have a jump in $\overline{x}$. Consequently,

$$D_{i_2(\overline{x})}(\mathcal{S}, \overline{x}) = D_{i_2(\overline{x})}(\mathcal{S}, \underline{x}) - i_2(\overline{x})(\overline{x} - \underline{x}).$$

This observation immediately leads to

$$D_{i_2(\underline{x})}(\mathcal{S}, \underline{x}) - D_{i_2(\overline{x})}(\mathcal{S}, \overline{x}) \geq D_{i_2(\overline{x})}(\mathcal{S}, \underline{x}) - D_{i_2(\overline{x})}(\mathcal{S}, \overline{x}) = i_2(\overline{x})(\overline{x} - \underline{x}).$$

By similar arguments we additionally obtain

$$D_{i_0(\underline{x})}(\mathcal{S},\underline{x}) - D_{i_0(\overline{x})}(\mathcal{S},\overline{x}) \leq D_{i_0(\underline{x})}(\mathcal{S},\underline{x}) - D_{i_0(\underline{x})}(\mathcal{S},\overline{x}) = i_0(\underline{x})(\overline{x}-\underline{x}).$$

Altogether we thus arrive at

$$\begin{aligned} f(\underline{x}) - f(\overline{x}) &= \big(D_{i_2(\underline{x})}(\mathcal{S},\underline{x}) - D_{i_2(\overline{x})}(\mathcal{S},\overline{x})\big) - \big(D_{i_0(\underline{x})}(\mathcal{S},\underline{x}) - D_{i_0(\mathcal{S},\overline{x})}(\overline{x})\big) \\ &\geq (i_2(\overline{x}) - i_0(\underline{x}))(\overline{x}-\underline{x}) \geq -\sigma_0(\overline{x}-\underline{x}) \end{aligned}$$

□

It is easy to check that, in addition to the properties of f which were stated so far, f does not have a jump in s_1. Although the significance of this observation is not revealed in the value for c^*, it has certain technical advantages. Let us now introduce the space of admissible functions $\mathscr{F}$ on the basis of everything that was discussed in this section.

Definition 2.14. A function $g : [0,1] \to \mathbb{R}$ is called *admissible* if it is subject to the following collection of properties.

(i) $g(0) = g(1) = 0$.

(ii) g is piecewise linear, piecewise monotonically decreasing, and $|g|$ is bounded by a^t.

(iii) g is left-continuous and each discontinuity appears as a jump of positive height.

(iv) The slope of g is always between $-a^t$ and $\sigma_0 := -a^{t-1}(a-2)$.

(v) If g is continuous on $[x,y]$ then the slope of $g(x)$ and $g(y)$ can differ by at most a^{t-1}.

(vi) There exists a set $\Gamma = \{\xi_1, \xi_2, \dots, \xi_{a^t-1}\} \subset [0,1)$ such that:

  a) If g has a jump in ξ then $\xi \in \Gamma$.

  b) There exists a set $\Gamma_1 \subset \Gamma$, $|\Gamma_1| = a^{t-1}(a-2)$, such that f has a jump of height at least one in each $\xi \in \Gamma_1$.

  c) There exist $a^{t-1}-1$ further points $\{\xi_{k_1}, \xi_{k_2}, \dots, \xi_{k_{a^{t-1}-1}}\} =: \Gamma_2$ with the following property:
  For each $1 \leq i < a^{t-1}$ let $\xi_{l_i}, \xi_{r_i} \in \Gamma \cup \{0,1\}$ such that $\Gamma \cap (\xi_{l_i}, \xi_{r_i}) = \{\xi_{k_i}\}$. Now, if there is an $\overline{x} \in (\xi_{l_i}, \xi_{r_i})$ with

$$\sigma(\overline{x}) > \sigma_0 - i \tag{2.3}$$

then

$$g(\underline{x}) \geq g(\overline{x}) - \sigma_0(\overline{x} - \underline{x}) \tag{2.4}$$

for all $\underline{x} \in [\xi_{l_i}, \xi_{k_i})$. Here, $\sigma(x)$ denotes the slope of g in x.

Furthermore, we call $\mathscr{F} = \{g : g \text{ is admissible}\}$ the *space of admissible functions*.

All the preceding paragraphs of this section may now be subsumed into the following central lemma below, cf. [48].

Lemma 2.15. *The function f as defined in Lemma 2.10 belongs to $\mathscr{F}$.*

From [46] it is known that the space of admissible functions $\mathscr{F}$ is closed with respect to pointwise convergence. Hence, we may deduce the existence of an admissible function f^* which is defined by the relation

$$\|f^*\|_1 := \min_{g \in \mathscr{F}} \|g\|_1. \tag{2.5}$$

Hence, with a view to Lemma 2.10, we aim for an estimate of the form $\|f^*\|_1 \geq b(a)$. This is dealt with in the next section.

2.2.3 A lower bound for the L^1-norm of the minimizer

Having the relation (2.5) at hand offers one significant advantage, namely, one can find a uniform lower bound for $\|f\|_1$ without knowing anything about the underlying sequence $\mathcal{S}$, simply because it is possible to make precise statements about the structure of the minimizer f^*.

In what follows we run through a series of lemmata which all serve to identify the exact shape of f^* step by step. We begin by showing that each discontinuity is enclosed by two zeros of f^*.

Lemma 2.16 (Cf. [46, Lemma 2.3], [48]Lemma 2)**.** *Let f^* have a discontinuity in γ. Then there exist two zeros of f^*, say, α and β with $\alpha < \gamma < \beta$, such that γ is the only discontinuity in the interval (α, β).*

Proof. If γ is the only point at which f^* has a jump, the claim is fulfilled with $\alpha = 0$ and $\beta = 1$. Hence it suffices to show the following statement: Let f^* have two successive discontinuities in, say, a_1 and a_2, $0 < a_1 < a_2 < 1$. Then f^* has a zero in the interval (a_1, a_2).

For contradiction we assume $f^* > 0$ on (a_1, a_2) (the case $f^* < 0$ can be treated quite similarly). In what follows, we construct an admissible function $\tilde{f}$ such that

$$\|\tilde{f}\|_1 < \|f^*\|_1,$$

which clearly contradicts the definition of f^*.

Naturally, we need to take special care in constructing $\tilde{f}$ if either $a_1 \in \Gamma_2$ or $a_2 \in \Gamma_2$ (see Definition 2.14). Moreover, if we manage to preserve the height of any existing jump in any other case then condition (vi.b) from this definition is automatically fulfilled for $\tilde{f}$.

First of all, we notice that f^* cannot have a kink at, say, $y \in (a_1, a_2)$ such that the slope before the kink is greater than afterwards. Indeed, let $\delta > 0$ such that the slope of f^* is constant on $[y-\delta, y)$ as well as on $(y, y+\delta]$. Then, as can be seen in Figure 2.2, we may interchange these pieces such that the resulting function $\tilde{f}$ (solid) remains continuous in $[y-\delta, y+\delta]$. Its absolute integral, however, is smaller than that of f^* (dashed). Thus, we need only

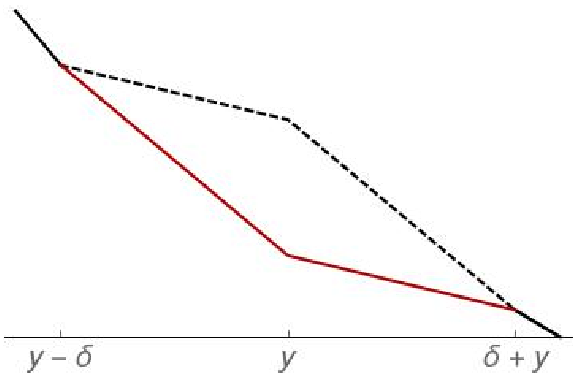

Figure 2.2: The slope of f^* may not get larger after a kink.

consider such kinks, where f^* becomes flatter.

Let now $a_2 \notin \Gamma_2$. We choose $\delta_1 > 0$ such that the slope of f^* is a constant σ_1 on $(a_2, a_2 + \delta_1)$. Furthermore, we set

$$\sigma = \min \{\sigma^*(x) : \ x \in (a_1, a_2 + \delta_1)\},$$

where σ^* denotes the slope of f^* and where we define $\sigma^*(a_2)$ as its left limit. Now, let $0 < \delta \leq \min\{-2f^*(a_2)/(\sigma_1 + \sigma), \delta_1\}$. With this choice of δ we have

$$f^*(a_2) + \sigma\delta > -f^*(a_2 + \delta).$$

In this case we may thus construct $\tilde{f}$ by moving the discontinuity to $\tilde{a}_2 = a_2 + \delta$. The missing part of $\tilde{f}$ on the left of $\tilde{a}_2$ of length δ is then chosen such that $\tilde{f}$ is continuous in a_2 and such that it has constant slope σ. This construction is visualized in Figure 2.3 (again f^* is represented by the dashed and $\tilde{f}$ by the solid line). This choice for the slope guarantees that the height of

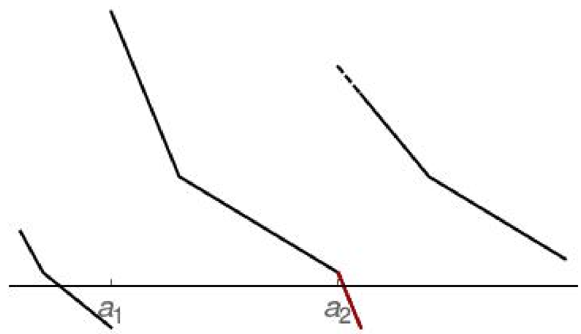

Figure 2.3: Alternative construction in the case $a_2 \notin \Gamma_2$.

the jump is preserved and, additionally, property (vi.c) from Definition 2.14, too, cannot be violated by this construction if $a_1 \in \Gamma_2$.

Certainly, the same construction also works if $a_2 = \xi_{k_i} \in \Gamma_2$ for a suitable k_i with $\sigma^* \leq -a^{t-1}(a-2) - i$ between a_2 and the next discontinuity of f^*.

However, if there is some point $x > a_2$ before the next jump of f^* with $\sigma^*(x) > -a^{t-1}(a-2) - i$ we have to proceed differently. In this case, we keep the discontinuity at a_2 and take the smallest such x, call it $\overline{x}$. We define

$$\tilde{f}(x) := \begin{cases} \sigma_0(\overline{x} - x) + f^*(\overline{x}) & \text{if } x \in [\overline{x} - \delta, \overline{x}), \\ \sigma^*(\overline{x})(\overline{x} - \delta - x) + \tilde{f}(\overline{x} - \delta) & \text{if } x \in [a_2, \overline{x} - \delta), \\ f^*(x) & \text{else,} \end{cases}$$

where $\delta > 0$ is such that we still have a positive jump in a_2. Recall that a discontinuity always constitutes a positive jump, hence this is possible. Figure 2.4 shows $\tilde{f}$ (solid) as well as f^* (dashed) in this case. The dotted line represents the line with maximal slope σ_0 reaching back from $\{\overline{x}, f^*(\overline{x})\}$ which occurs in Definition 2.14. Notice that, again,

$$\|\tilde{f}\|_1 < \|f^*\|_1$$

and that (vi.c) from Definition 2.14 is not violated for a_2. Additionally, the condition on δ guarantees that (vi.c) is not violated for a_1 if $a_1 \in \Gamma_2$ either. Moreover, we need not take care of the height of the jump in a_2, since Γ_1 and Γ_2 are disjoint. □

Thus, f^* consists of parts Q, each of which is defined on an interval $[\alpha, \beta]$ with $f^*(\alpha) = f^*(\beta) = 0$ and such that there is exactly one discontinuity in (α, β), see Figure 2.5. In the following lemma we determine the number of such Q's for f^*.

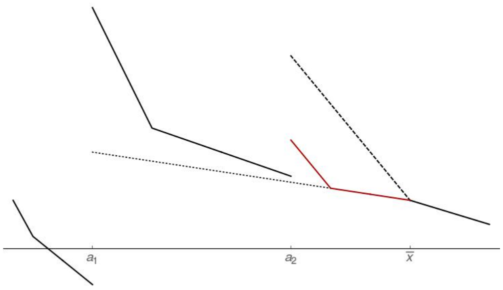

Figure 2.4: Alternative construction in the case $a_2 \in \Gamma_2$.

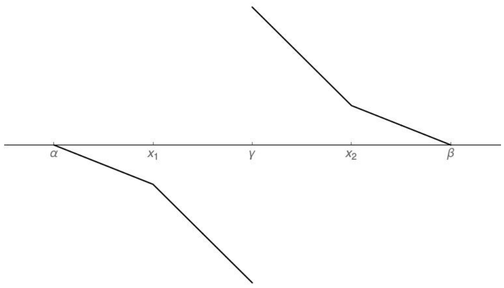

Figure 2.5: An exemplary plot of a part Q.

Lemma 2.17 (Cf. [46, Lemma 2.4], [48, Lemma 3]). *The function f^* has exactly $a^t - 1$ discontinuities.*

Proof. Assume that the total number of discontinuities of f^* is strictly less than $a^t - 1$. Then we define a strictly admissible function $\tilde{f}$ from f^* whose absolute integral is smaller than that of f^*. To this end, let Γ^* be the set Γ from property (vi) for the function f^*.

By assumption there is a $\xi^* \in \Gamma^*$ such that f^* is continuous in ξ^*. The definition of Γ_1^* (i.e., the set Γ_1 for f^*) guarantees $\xi^* \notin \Gamma_1^*$. Let us confine ourselves to the case where $\xi^* \in \Gamma_2^*$. The case $\xi^* \in \Gamma_0^* := \Gamma^* \setminus (\Gamma_1^* \cup \Gamma_2^*)$ can be treated analogously.

Here, we choose $\gamma \in \Gamma^*$ such that f^* has a jump in γ. We show that $\gamma \in \Gamma_1^*$ and that f^* has a jump of height 1 in γ (case (d) below). Indeed, à priori we are in one of the following four cases:

(a) $\gamma \in \Gamma_2^*$,

(b) $\gamma \in \Gamma_0^*$,

(c) $\gamma \in \Gamma_1^*$ with a jump of height greater than 1, or

(d) $\gamma \in \Gamma_1^*$ with a jump of height exactly equal to 1 in γ.

Assume that $\gamma \in \Gamma_2^*$ (case (a)). By Lemma 2.16 γ is isolated by two successive zeros of f^*. Hence, (2.4) from property (vi) cannot hold, and therefore (2.3) from the same property does not hold either. Consequently, we can take a point $\tilde{\xi}$ to the left of γ and insert a short piece of minimal slope on $[\tilde{\xi}, \gamma)$ without interferring with property (vi.c), see Figure 2.6. Again, the dashed line represents f^* and the solid one the constructed function $\tilde{f}$. The new set $\tilde{\Gamma}$ is the set Γ^* with ξ^* replaced by $\tilde{\xi}$.

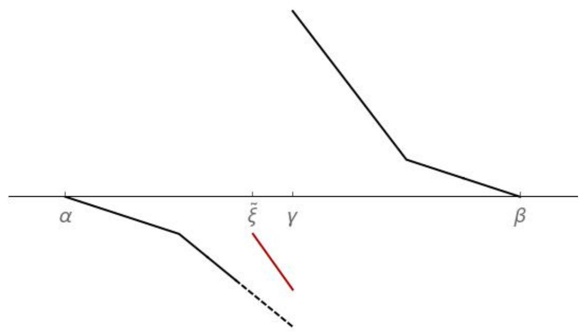

Figure 2.6: Alternative construction if $\gamma \in \Gamma_2^*$.

This construction works in the same way for case (b), and, with some special care, i.e. the jump of $\tilde{f}$ in γ maintains a height of at least one, for case (c) too.

Consequently, f^* can only have $a^{t-1}(a-2)$ jumps at the positions given by Γ_1^*. All these jumps have height exactly equal to one and there are absolutely no further discontinuities. Obviously, f^* cannot have slope $-a^t$ everywhere, since then

$$0 > a^{t-1}(a-2) - a^t = f^*(1),$$

a contradiction to property (i). Thus, there exists an interval $[\delta_1, \delta_2]$ such that $f^* > 0$ (or $f^* < 0$) on $[\delta_1, \delta_2]$ and its slope is greater than $-a^t$. We choose $\delta' \in (\delta_1, \delta_2)$ sufficiently close to δ_1 (or to δ_2) and define

$$\tilde{f}(x) = \begin{cases} f^*(\delta_1) - a^t(x - \delta_1) & \text{if } x \in (\delta_1, \delta'], \\ f^*(x) & \text{else,} \end{cases}$$

or

$$\tilde{f}(x) = \begin{cases} f^*(\delta_2) - a^t(x - \delta_2) & \text{if } x \in (\delta', \delta_2], \\ f^*(x) & \text{else,} \end{cases}$$

respectively. See Figures 2.7 and 2.8. Clearly, $f \in \mathscr{F}$ and once again we have $\|\tilde{f}\|_1 < \|f^*\|_1$, a contradiction.

□

From the above results we see that the shape of f^* can be characterized in the following way: The function f^* divides $[0, 1)$ into $a^t - 1$ parts $[\alpha, \beta)$ with $f^*(\alpha) = f^*(\beta) = 0$, and, on each such part, f^* has exactly one discontinuity $\gamma \in (\alpha, \beta)$. We say that $[\alpha, \beta)$ is of type Q_j iff $\gamma \in \Gamma_j^*$ for $j = 0, 1, 2$. The aim of the subsequent paragraphs is to determine the shape of these parts more clearly. For parts of the classes Q_0 and Q_1 this has already been done in [46], the remaining class Q_2 has been dealt with in [69].

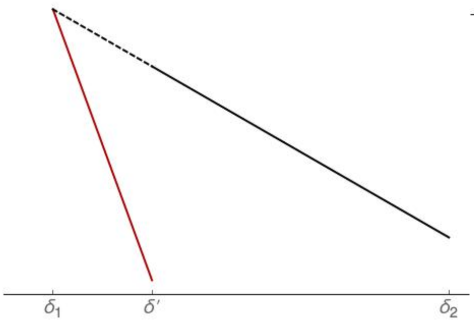

Figure 2.7: Case $f^*>0$ on $[\delta_1,\delta_2]$.

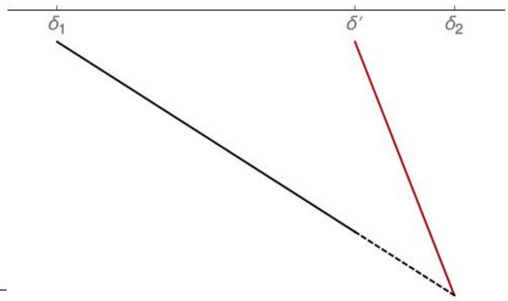

Figure 2.8: Case $f^*<0$ on $[\delta_1,\delta_2]$.

Lemma 2.18 (Characterization of Q_0, cf. [46, (2)],)**.** *Let f^* be as defined in 2.5 and let $\alpha < \beta$ denote the zeros of f^*. Then*

$$\int_\alpha^\beta |f^*(x)|\,\mathrm{d}x \geq \frac{|\sigma_0|}{4}\chi_0^2, \qquad \chi_0 = \beta - \alpha.$$

Proof. It follows from simple calculations that the smallest possible configuration in this setting is taking a function g with $g(\alpha) = g(\beta) = 0$, a jump in $\gamma = (\alpha+\beta)/2$ and slope $\sigma_0 = -a^{t-1}(a-2)$ everywhere, i.e., the largest possible slope for admissible functions. Notice that property (vi.c) from Definition 2.14 is not necessarily fulfilled for g. Nevertheless, the integral of $|f^*|$ is greater than that of $|g|$, which is given by the right-hand side in the claim. □

As a matter of fact, there exists a configuration such that the function g from the above proof would be admissible (simply do not let parts Q_0 be followed by parts Q_2). However, it would be much harder to show that such a configuration actually yields a minimal L^1-norm.

Lemma 2.19 (Characterization of Q_1, cf. [46, Lemmata 2.6–2.9])**.** *Let f^* be as defined in (2.5) and let $\alpha < \beta$ denote the zeros and $\gamma \in (\alpha, \beta)$ the discontinuity of f^* in Q_1. Furthermore, we set*

$$-\delta = f^*(\gamma) \qquad \textit{and} \qquad \tau = \lim_{x\downarrow\gamma} f^*(x).$$

Then the following statements hold.

(a) For every fixed configuration $(\alpha, \beta, \gamma, \delta, \tau)$ there are unique points $x_1 \in [\alpha, \gamma]$ and $x_2 \in [\gamma, \alpha]$ which define an admissible function $\tilde{f}_{v,v'}$, $v, v' \in [0,2]$ by:

- $\tilde{f}_{v,v'}(\alpha) = \tilde{f}(\beta) = 0$,
- $\tilde{f}_{v,v'}(\gamma) = -\delta$, $\lim_{x\downarrow\gamma} \tilde{f}_{v,v'}(x) = \tau$,
- $\tilde{f}_{v,v'}$ *has slope*
$$\sigma_{\min}(v) = -(a^t - va^{t-1})$$
in $[x_1, \gamma]$ *and* $\sigma_{\min}(v')$ *on* $[\gamma, x_2]$*, as well as*
- *slope*
$$\sigma_{\max}(v) = \min\left\{-(a^t - (v+1)a^{t-1}), \sigma_0\right\}$$
in $[\alpha, x_1]$ *and* $\sigma_{\max}(v')$ *on* $[x_2, \beta]$.

(b) *The function* f^* *is of the form* $\tilde{f}_{v,v'}$ *with* $0 \le v, v' \le 1$ *and the height of the jump is exactly 1, i.e.* $\delta + \tau = 1$.

(c) f^* *has its discontinuity at* $\gamma = (\alpha+\beta)/2$ *and its absolute integral equals*
$$\int_\alpha^\beta |f^*(x)|\,\mathrm{d}x = \frac{\chi_1(4 - a^{t-1}\chi_1)}{16}, \qquad \chi_1 = \beta - \alpha.$$

Proof. Straightforward calculations yield
$$x_1 = \frac{-\delta + \alpha\sigma_{\max}(v) - \gamma\sigma_{\min}(v)}{\sigma_{\max}(v) - \sigma_{\min}(v)} \quad \text{as well as} \quad x_2 = \frac{\tau - \gamma\sigma_{\min}(v') + \beta\sigma_{\max}(v')}{\sigma_{\max}(v') - \sigma_{\min}(v')}$$
and, hence, the first item (a) is verified.

For the second item (b) we confine ourselves to the interval $[\alpha, \gamma]$ and denote the minimal slope of f^* on $[\alpha, \gamma]$ by $\sigma^*_{\min}(v) = -(a^t - va^{t-1})$ for some $v \in [0, 2]$. Due to condition (v) we know that the maximal slope of f^*, $\sigma^*_{\max}(v)$, can differ by at most a^{t-1}, i.e.
$$\sigma^*_{\max} \le \min\left\{-(a^t - (v+1)a^{t-1}), \sigma_0\right\}.$$
This immediately implies $f^* \ge \tilde{f}_{v,v'}$ on $[\alpha, \gamma]$. This together with similar observations concerning the interval $(\gamma, \beta]$ thus lead to
$$\int_\alpha^\beta |f^*(x)|\,\mathrm{d}x \ge \int_\alpha^\beta |f_{v,v'}(x)|\,\mathrm{d}x$$
and, consequently, $f^* = \tilde{f}_{v,v'}$ on $[\alpha, \beta]$ for some $v, v' \in [0, 2]$. Additionally, it can be seen that the absolute integral is larger for $v > 1$ (or $v' > 1$) than for $v = 1$ (or $v' = 1$, resp.), hence we need only consider $v, v' \in [0, 1]$. To finish the proof of item (b) it remains to show that $\delta + \tau = 1$. For contradiction we assume $\delta + \tau > 1$. Let v_0, v_0' be the parameters such that $f^* = \tilde{f}_{v_0,v_0'}$.

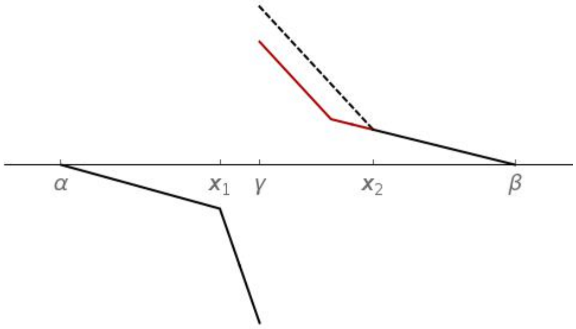

Figure 2.9: Alternative construction if $\delta + \tau > 1$.

In this case we construct a function $g_{v_0,v_0'}$ of the same kind (in the sense of (a)) and the same slope parameters v_0, v_0' but with $g(\gamma) = -1 + \tau > \delta$. This construction is depicted in Figure 2.9. The function $g_{v_0,v_0'}$ is again admissible and its absolute integral over $[\alpha, \beta]$ is clearly smaller than that of f^*, a contradiction.

Let us now focus on item (c) of the claim. We compute $\int_\alpha^\gamma |\tilde{f}_{v,v'}(x)|\,\mathrm{d}x$ as well as $\int_\gamma^\beta |\tilde{f}_{v,v'}(x)|\,\mathrm{d}x$ and notice that the resulting expressions are quadratic polynomials in v and v', respectively, whose minima are attained at

$$v_0 = a - \frac{1}{2} - \frac{\delta}{a^{t-1}(\gamma - \alpha)} \qquad \text{and} \qquad v_0' = a - \frac{1}{2} - \frac{1-\delta}{a^{t-1}(\beta - \gamma)}.$$

Subsequently, we consider the entire integral with the above parameters inserted, i.e. $\int_\alpha^\beta |\tilde{f}_{v_0,v_0'}(x)|\,\mathrm{d}x$, and minimize with respect to γ, giving

$$\gamma_0(\delta) = \frac{\alpha + \beta}{2} - \frac{1 - 2\delta}{a^{t-1}}.$$

Again, this entails minimizing a quadratic polynomial with positive leading coefficient. The same procedure (with $\gamma_0(\delta)$ inserted into the integral) finally yields $\delta = 1/2$. Consequently,

$$\gamma_0 = \frac{\alpha+\beta}{2}, \quad v = v' = -\frac{1}{2} + a - \frac{1}{a^{t-1}(\beta - \alpha)}, \quad \text{and} \tag{2.6}$$

$$\int_\alpha^\beta |\tilde{f}_{v_0,v_0'}(x)|\,\mathrm{d}x = \int_\alpha^\beta |f^*(x)|\,\mathrm{d}x = \frac{(\beta-\alpha)\left(4 - a^{t-1}(\beta - \alpha)\right)}{16}. \tag{2.7}$$

□

Finding the exact shape of parts Q_2 is again a relatively simple task as can be seen in the lemma below.

Lemma 2.20 (Characterization of Q_2, cf. [48, Lemma 4]). *The parts Q_2 can be subdivided further into parts $Q_2^{(i)}$, $1 \leq i < a^{t-1}$. Let the corresponding zeros of f^* be denoted by $\alpha_i < \beta_i$ and let γ_i be the unique discontinuity of f^* on $[\alpha_i, \beta_i]$. Then f^* has slope σ_0 on $[\alpha_i, \gamma_i)$ and slope $\sigma_0 - i$ on $(\gamma_i, \beta_i]$. Moreover,*

$$\gamma_i = \frac{|\sigma_0|(\alpha_i + \beta_i) + i\beta_i}{i + 2|\sigma_0|}, \qquad \text{and} \qquad \int_{\alpha_i}^{\beta_i} |f^*(x)| \, \mathrm{d}x = \left(\chi_2^{(i)}\right)^2 \frac{|\sigma_0|(i + |\sigma_0|)}{2(i + 2|\sigma_0|)},$$

where $\chi_2^{(i)} = \beta_i - \alpha_i$.

Proof. Evidently, the inequality (2.4) cannot hold due to Lemma 2.16 and the fact that f^* has negative slope (cf. Definition 2.14). Since f^* is admissible, however, (2.3) cannot hold either. This means that $\sigma^* \leq \sigma_0 - i$ on $(\gamma_i, \beta_i]$, where σ^* denotes the slope of f^*. Clearly, $\sigma^* = \sigma_0$ on $[\alpha_i, \gamma_i)$ and $\sigma^* = \sigma_0 - i$ on $(\gamma_i, \beta_i]$. So it remains to determine γ_i. To this end we once again compute the absolute integral of f^* over $[\alpha_i, \beta_i]$, which turns out to be a quadratic polynomial in γ_i with positive leading coefficient. The minimum is attained at

$$\gamma_i = \frac{|\sigma_0|(\alpha_i + \beta_i) + i\beta_i}{i + 2|\sigma_0|},$$

where we have

$$\int_{\alpha_i}^{\beta_i} |f^*(x)| \, \mathrm{d}x = (\beta_i - \alpha_i)^2 \frac{|\sigma_0|(i + |\sigma_0|)}{2(i + 2|\sigma_0|)}.$$

□

We are now in a position to derive a uniform (w.r.t. t) lower bound for $\|f^*\|_1$ and thus finalize the proof of Theorem 2.9 with the following lemma.

Lemma 2.21 (Cf. [48, Lemma 5]). *For all $3 \leq a \leq 3.7$ we have*

$$\|f^*\|_1 \geq \frac{(a-2)\left(12a + 9 + (a-2)(4a-3)\log\left(1 + \frac{1}{a-2}\right)\right)}{16\left(a - \frac{1}{2}\right)^2 \left(3 + (a-2)\log\left(1 + \frac{1}{a-2}\right)\right)}.$$

Proof. We begin by showing that all the parts of type Q_0 are of the same length (cf. [46, Lemma 2.10]). Indeed, let Q_0' and Q_0'' be two such parts with lengths χ_0' and χ_0'', respectively, and assume $\overline{\chi} = \chi_0' + \chi_0''$. Now, Lemma 2.18 implies

$$\int_{Q_0' \cup Q_0''} |f^*(x)| \, \mathrm{d}x = \frac{|\sigma_0|}{4} \left((\chi_0')^2 + (\chi_0'')^2\right).$$

Simple calculations show that the left-hand side attains its minimum at $\chi_0' = \chi_0'' = \overline{\chi}$. In the same spirit one can derive an analogous statement for parts of type Q_1.

Let us now focus on the lower bound from the claim. To this end we denote the lengths of parts of type Q_0 and Q_1 by χ_0 and χ_1, respectively. Furthermore, let $\chi_2^{(i)}$ be the corresponding lengths of the parts of type $Q_2^{(i)}$, which are defined in Lemma 2.20. Due to the above discussion and Lemma 2.18, Lemma 2.19, and Lemma 2.20 we have to minimize the right-hand side of

$$\begin{aligned}\|f^*\|_1 \geq a^{t-1} \cdot \chi_0^2 \frac{a^{t-1}(a-2)}{4} + a^{t-1}(a-2) \cdot \frac{\chi_1\,(4-a^{t-1}\chi_1)}{16} \\ + \sum_{i=1}^{a^{t-1}-1} \left(\chi_2^{(i)}\right)^2 \frac{|\sigma_0|(i+|\sigma_0|)}{2(i+2|\sigma_0|)} \\ =: a^{t-1} \cdot \chi_0^2 \tilde{A}_0 + a^{t-1}(a-2) \cdot \frac{\chi_1\,(4-a^{t-1}\chi_1)}{16} + \sum_{i=1}^{a^{t-1}-1} \left(\chi_2^{(i)}\right)^2 \tilde{A}_i\end{aligned}$$

with respect to $\chi_0, \chi_1, \chi_2^{(i)} \geq 0$ under the constraint

$$a^{t-1}\chi_0 + a^{t-1}(a-2)\chi_1 + \sum_{i=1}^{a^{t-1}-1} \chi_2^{(i)} = 1.$$

With the Lagrangian approach we immediately obtain

$$\tilde{A}_0\chi_0 = \tilde{A}_i\chi_2^{(i)} \quad \text{for all } \ 1 \leq i < a^{t-1}.$$

The constraint is therefore equivalent to

$$\chi_0 = \frac{1-a^{t-1}(a-2)\chi_1}{a^{t-1} + \sum_{i=1}^{a^{t-1}-1} \frac{\tilde{A}_0}{\tilde{A}_i}}.$$

Moreover, the denominator in the above equation simplifies to

$$\begin{aligned}a^{t-1} + \sum_{i=1}^{a^{t-1}-1} \frac{\tilde{A}_0}{\tilde{A}_i} &= a^{t-1} + \sum_{i=1}^{a^{t-1}-1} \left(1 - \frac{i}{2(|\sigma_0|+i)}\right) \\ &= 2a^{t-1} - 1 - \frac{1}{2} \sum_{i=|\sigma_0|+1}^{a^{t-1}-1+|\sigma_0|} \left(1 - \frac{|\sigma_0|}{i}\right) \\ &= \frac{1}{2}\left(3a^{t-1} - 1 + |\sigma_0| \sum_{i=|\sigma_0|+1}^{a^{t-1}-1+|\sigma_0|} \frac{1}{i}\right).\end{aligned}$$

The latter sum can be bounded by $\log(1+1/(a-2))$ from above. We summarize our intermediate findings and obtain

$$\|f^*\|_1 \geq \frac{(a-2)\left(1-a^{t-1}(a-2)\chi_1\right)^2}{2\left(3+(a-2)\log\left(1+\frac{1}{a-2}\right)\right)} + a^{t-1}(a-2)\frac{\chi_1(4-a^{t-1}\chi_1)}{16} =: p(\chi_1).$$

In what follows, our goal is to minimize the function p. We immediately see that p is a polynomial of degree two and its leading coefficient is positive for all $3 < a \leq 3.7$. Thus, it attains its minimum at its only critical point

$$\chi_{\text{crit}} = a^{1-t}\frac{2\left(4a-11-(a-2)\log\left(1+\frac{1}{a-2}\right)\right)}{29+8a(a-4)-(a-2)\log\left(1+\frac{1}{a-2}\right)}.$$

On the other hand (cf. [46]), we know from (2.6) and the proof of Lemma 2.19 that

$$0 \leq v_0 = -\frac{1}{2} + a - \frac{1}{a^{t-1}\chi_1} \leq 1.$$

Hence we have the following bounds for χ_1

$$\chi_{\min} := \frac{a^{1-t}}{a-\frac{1}{2}} \leq \chi_1 \leq \frac{a^{1-t}}{a-\frac{3}{2}}.$$

We finish the proof by showing that $\chi_{\text{crit}} \leq \chi_{\min}$. Indeed, it can easily be verified that the denominator of χ_{crit} is positive. Thus, $\chi_{\text{crit}} > \chi_{\min}$ if and only if

$$0 > 3a - 9 - (a-1)(a-2)\log\left(1+\frac{1}{a-2}\right) =: \tilde{p}(a).$$

Furthermore, we observe that $\tilde{p}(3.7) < 0$ and, additionally, that $\tilde{p}'(a) > 0$ for all $a \in (3, 3.7]$. Hence

$$\chi_1 = \frac{a^{1-t}}{a-\frac{1}{2}}$$

and by inserting this value into the function p the result immediately follows.

□

2.2.4 Discussion and open questions

Recapitulating the previous Section 2.2.3 we notice that we actually do have a lot of information on the minimizer f^* at hand. Indeed, Lemmata 2.18–2.21 very clearly indicate its structure. The only thing that deprives the proof from being *"constructive"* is Lemma 2.18. As it was already hinted in the paragraph before Lemma 2.19, problems might occur if parts of type

Q_2 are succeeded by parts of type Q_0. Indeed, in this case f^* experiences a kink where its slope becomes flatter at the transition between the two parts. This clearly is in conflict with item (vi.c) from Definition 2.14. Nevertheless, there exists a configuration, where this problem is taken into account: simply let each part of type Q_2 be followed by a part Q_1 or be placed at the very end of $[0, 1)$. Summarizing, we have thus determined the shape of several minimizers by pairing parts Q_2 with parts Q_1 (with one possible exception at the end of $[0, 1)$) and filling the entire unit interval with these pairs, a^{t-1} parts of type Q_0, and the remaining parts of type Q_1.

Let us now consider the *inverse* to the minimization problem that has just been solved, i.e., find a sequence $\mathcal{S}$ such that the corresponding discrepancy function $D_N(\mathcal{S}, \cdot)$ is of (almost) the same form as f^*. Of course, it is not clear at all whether such a configuration even exists. Nevertheless, we know that parts of type Q_j correspond to elements of the sequence $\mathcal{S}$ indexed by A_j, $0 \leq j \leq 2$. Furthermore, we determined where this element is located relative to the length of the interval on which the underlying part Q_j is defined (i.e., the γ from within the respective characterization lemma of Q_j). Together with the structural restriction from the above paragraph one might thus hope to be able to, first of all, construct a sequence for which we obtain a discrepancy bound comparable to $c^* \log N$ and, secondly, maybe even discover a new class of low-discrepancy sequences.

We take the above discussion as an incentive to formulate the following open problem related to this chapter.

Open Problem 2.22. Is it possible to extract a construction principle for low-discrepancy sequences from the detailled information on f^* that is gathered within Section 2.2.3?

2.3 Point sets in the unit cube

Within this section we tackle the famous Theorem 2.5 by Bilyk and Lacey from 2008, see [11]. To be more precise, we build upon their methods, give full details where necessary, and improve certain steps to quantify their result. This outcome is made precise in the theorem below (see [69, Theorem 1]).

Theorem 2.23. *For all $\epsilon > 0$ and all N-point sets $\mathcal{P}$ in $[0, 1)^3$ there exists an absolute constant $C > 0$ such that*

$$D_N^*(\mathcal{P}) \geq C(\log N)^{1+\eta}, \qquad \text{where} \qquad \eta = \frac{1}{32 + 4\sqrt{41}} - \epsilon,$$

for all N sufficiently large.

To approach the proof of this theorem there are several main tasks that need to be taken care of. First of all, the relevant tools and methods from harmonic analysis have to be set out. To this end, we present Halász' proof of Schmidt's result Theorem 2.2 for point sets in two dimensions in order to get the reader familiarized with the Haar function system and the underlying strategy of the proof. Furthermore, this shall serve as an illustration of as well as a hint to why the very same proof does not work in dimensions three or higher. All this is covered in Section 2.3.1.

As it has already been explained in the introduction, Halász' methods rely on certain orthogonality arguments, which we lack in dimensions $d \geq 3$, and other Hilbert space specific features. Hence, we require new techniques, which come in form of a generalization of Parseval's identity to L^p-spaces, $1 < p < \infty$, the so-called *Littlewood–Paley inequalities*. These are explained in full detail and discussed with a view to application within our proof in Section 2.3.2.

The main ingredient that Halász contributed to Bilyk's and Lacey's proof is the use of an auxiliary function of a certain form. More precisely, he employed a Riesz product composed of sums of Haar functions and considered its inner product with the discrepancy function. Section 2.3.3 introduces the Riesz product tailored to our problem that has been used in [11]. Furthermore, this part of the thesis also contains the main Lemma 2.31 which collects the most important results from Section 2.3 and immediately allows for a proof of Theorem 2.23.

Section 2.3.4 focuses on the simplest instance of the above mentioned shortfall of orthogonality properties and deals with them using the Littlewood–Paley inequalities as introduced in Section 2.3.2. This, in turn, vastly faciliates the proof of several items of our central Lemma 2.31, as can be seen in Section 2.3.5 and, eventually, provides an upper bound of the aforementioned inner product.

Subsequently, we tackle the orthogonality problems, which were mentioned above, again in Section 2.3.6, but this time for more complicated instances as those confronted in Section 2.3.4. This is done by incorporating the ideas of Beck (cf. [4]) and extensive applications of the Littlewood–Paley inequalities. As a result we obtain the last missing item from our main lemma, which also accounts for the critical value we obtain for the exponent η in Theorem 2.23.

In the end we still require a lower bound for the inner product of our auxiliary function with the discrepancy function. As we will see in Section 2.3.7, this can be achieved via a certain choice of yet still free parameters within our Riesz product.

Finally, we include a brief discussion and a selection of open problems related to our findings in Section 2.3.8 to conclude the study of arbitrary point sets in three dimensions.

2.3.1 Preliminaries and Halász' proof of Theorem 2.2

Within this section we show Schmidt's bound (Theorem 2.2) for point sets in the unit square, i.e., for all N-point sets $\mathcal{P} \subseteq [0,1)^2$ we have

$$D_N^*(\mathcal{P}) \geq C \log N \tag{2.8}$$

for an absolute constant $C > 0$ and all sufficiently large N.

The essential idea behind the proof of Halász is to choose an auxiliary function Φ in such a way that it is complicated enough to recapture the overall structure of $D_N(\mathcal{P}, \cdot)$ well, while, on the other hand, it behaves nicely in average. More precisely, one constructs Φ such that $\|\Phi\|_1 \leq 2$ and $\langle \Phi, D_N(\mathcal{P}, \cdot)\rangle \geq 2C \log N$ for some $C > 0$ since then, by duality,

$$2D_N^*(\mathcal{P}) = 2\|D_N(\mathcal{P}, \cdot)\|_\infty \geq \langle \Phi, D_N(\mathcal{P}, \cdot)\rangle \geq 2C \log N. \tag{2.9}$$

We pursue this idea by adapting the proof given by Matoušek in [54, Section 6.2] to the notation used in [11] and [69], as it was done in [8]. The desired behaviour of Φ can be achieved by using sums of Haar functions equipped with some sign. It needs to be mentioned that already Roth's celebrated Theorem 2.1 relies on auxiliary functions constructed from sums of Haar functions.

Definition 2.24 (Haar functions)**.** Let $\mathcal{D}$ denote the class of dyadic intervals, i.e.

$$\mathcal{D} = \{[a2^{-k}, (a+1)2^{-k}) : \ k \in \mathbb{N} \text{ and } 0 \leq a < 2^k\}.$$

Furthermore, we subdivide each $J \in \mathcal{D}$ into a left and a right half, J_l and J_r, respectively, and define the *one-dimensional Haar function* as $h_J = -\mathbb{1}_{J_l} + \mathbb{1}_{J_r}$. In higher dimensions $d \geq 2$ we take a dyadic rectangle $R = J_1 \times J_2 \times \cdots \times J_d \in \mathcal{D}^d$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_d) \in [0,1)^d$ and set

$$h_R(\boldsymbol{x}) = h_{J_1}(x_1) h_{J_2}(x_2) \cdots h_{J_d}(x_d). \tag{2.10}$$

One of the main advantages of working in this function system is that products of Haar functions yield Haar functions again in some cases. This is indicated in the following proposition, see [11, Proposition 6.1].

Proposition 2.25 (Product rule)**.** Let $R_1, R_2, \ldots, R_k \in \mathcal{D}^d$ be a collection of dyadic rectangles with non-empty intersection. Moreover, let us denote by $R_j^{(t)}$ the t-th coordinate of the rectangle R_j and assume that $R_1^{(t)}, R_2^{(t)}, \ldots, R_k^{(t)}$ are mutually different (not disjoint) for each $1 \leq t \leq d$. Then we have

$$h_{R_1} h_{R_2} \cdots h_{R_k} = \sigma h_S, \quad \text{where } S = R_1 \cap \cdots \cap R_k \text{ and } \sigma \in \{-1, +1\}.$$

Proof. We take an arbitrary $\boldsymbol{x} = (x_1, x_2 \ldots, x_s) \in [0,1)^d$ and expand the above product, giving

$$\prod_{j=1}^{k} h_{R_j}(\boldsymbol{x}) = \prod_{j=1}^{k} \prod_{t=1}^{d} h_{R_j^{(t)}}(x_t).$$

As $S = (S_1, S_2, \ldots, S_d)$ is defined as the intersection of all rectantgles involved and since all these rectangles are distinct in each coordinate there exists a unique k_0 for each t with $S_t = R_{k_0}^{(t)}$. Observe that $h_{R_j^{(t)}}|S_t =: \sigma_{j,t}$ is constantly either -1 or $+1$ for all $j \neq k_0$. Thus, we have

$$\prod_{j=1}^{k} h_{R_j}(\boldsymbol{x}) = \prod_{t=1}^{s} h_{S_t}(x_t) \prod_{j=1}^{k} \sigma_{j,t} = \sigma h_S(x), \quad \sigma \in \{-1, +1\}.$$

□

It is an immediate observation that the mean of a dyadic Haar function equals zero. Hence, the product rule accounts for orthogonality in some sense.

We are particularly interested in collections of so-called *hyperbolic* dyadic rectangles, i.e. a collection where all rectangles share the same volume. On the basis of these we introduce certain linear combinations of Haar functions which are the main building blocks of the auxiliary functions in both cases $d = 2, 3$. All this is covered within the definition below.

Definition 2.26. For $n \in \mathbb{N}$ let

$$\mathbb{H}_n^d = \left\{ \vec{r} = (r_1, r_2, \ldots, r_d) \in \mathbb{N}^d : \ \|\vec{r}\|_{\ell^1} = n \right\}.$$

Here, the letter "$\mathbb{H}$" is used to resemble the term *hyperbolic*. Furthermore, we call a collection of k hyperbolic vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_k \in \mathbb{H}_n^d$ *strongly distinct* iff for all coordinates $1 \leq t \leq d$ the numbers $r_{1,t}, r_{2,t}, \ldots, r_{k,t}$, $\vec{r}_j = (r_{j,1}, r_{j,2}, \ldots, r_{j,d})$, are mutually distinct. If two or more of these vectors fail to be strongly distinct we say that they have a *coincindence*.

Now, let $\vec{r} \in \mathbb{H}_n^d$ for some n and

$$\mathcal{D}_{\vec{r}} = \left\{ R = (J_1, J_2, \ldots, J_d) \in \mathcal{D}^d : \ |J_t| = 2^{-r_t} \right\}. \tag{2.11}$$

We call the function

$$f_{\vec{r}} = \sum_{R \in \mathcal{D}_{\vec{r}}} \alpha(R) h_R, \qquad \alpha(R) \in \{-1, 1\} \tag{2.12}$$

an r-*function with parameter* $\vec{r} \in \mathbb{H}_n^d$.

At this point some remarks are in order.

Remark 2.27. (i) Notice that $\int_0^1 h_R(x)\,\mathrm{d}x = 0$ for any $R \in \mathcal{D}$. Together with Proposition 2.25 and the product structure of higher-dimensional Haar functions (2.10) this immediately implies that, for $R_1, R_2, \ldots, R_k \in \mathcal{D}^d$, we have

$$\int_{[0,1)^d} h_{R_1}(\boldsymbol{x}) h_{R_2}(\boldsymbol{x}) \cdots h_{R_k}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0$$

if $\min\{|R_{j,t}| : 1 \le j \le k\}$ is unique for some $1 \le t \le d$, where $R_{j,t}$ denotes the t-th coordinate of the rectangle R_j.

(ii) In two dimensions the concept of strong distinctiveness is rendered obsolete by the hyperbolic assumption. Indeed, if we fix $\vec{r} = (r_1, n - r_1)$ then *every* different vector of the form $\vec{s} = (s_1, n - s_2)$ is automatically different in *each* coordinate.

(iii) Let $R_1, R_2 \in \mathcal{D}_{\vec{r}}$, $R_1 \neq R_2$, for some $\vec{r} \in \mathbb{H}_n^d$. Then we necessarily have $R_1 \cap R_2 = \emptyset$ and hence $f_{\vec{r}}^2 = \mathbb{1}_{[0,1)^d}$.

(iv) In some cases the product of two r-functions $f_{\vec{r}}$ and $f_{\vec{s}}$ is an r-function again. Indeed, their product can be expanded into a sum of products of Haar functions (up to signs). The product of Haar functions, in turn, is again a Haar function if their supporting rectangles meet the prerequisites of Proposition 2.25, that is, if $\vec{r}$ and $\vec{s}$ are strongly distinct.

(v) Considering item (i) in (iv) we see that

$$\int_{[0,1)^d} f_{\vec{r}_1}(\boldsymbol{x}) f_{\vec{r}_2}(\boldsymbol{x}) \cdots f_{\vec{r}_k}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0$$

if $\max\{r_{j,t} : 1 \le j \le k\}$ is unique for some coordinate $1 \le t \le d$, where $r_{j,t}$ denotes the t-th coordinate of $\vec{r}_j$.

The last but one item from the above remark probably captures the main difference between the two-dimensional and three-dimensional setting best. In the case $d = 2$ the product of two different r-functions always yields an r-function and therefore has mean zero. In dimension three already this argument cannot be repeated verbatim. We thus need to consider strongly distinct hyperbolic vectors or, at least, refer to item (v) to obtain mean zero. Non-surprisingly, a great amount of effort is therefore dedicated to the analysis of coincidences, see Sections 2.3.4 and 2.3.6.

Let us now tackle the proof of (2.8). To this end, we choose n such that $2^{n-2} \leq N < 2^{n-1}$ and for each $0 \leq k \leq n$ we define the r-functions $f_k := f_{(k,n-k)}$ with a not yet specified choice of signs. Furthermore, we fix a weight $\gamma \in (0,1)$ and define Φ as the following Riesz product below

$$\Phi = \prod_{k=0}^{n} (1 + \gamma f_k) - 1. \tag{2.13}$$

After expanding the product we find that Φ can be rewritten in the form $\Phi = \Phi_1 + \Phi_2 + \cdots + \Phi_n$ with

$$\Phi_k = \gamma^k \sum_{0 \leq j_1 < j_2 < \cdots < j_k \leq n} f_{j_1} f_{j_2} \cdots f_{j_k}. \tag{2.14}$$

Clearly, Φ is bounded in the L^1-norm, since

$$\|\Phi\|_1 \leq 1 + \int_{[0,1)^2} \prod_{k=1}^{n} (1 + \gamma f_k(\boldsymbol{x}))\, \mathrm{d}\boldsymbol{x} = 2 + \sum_{k=1}^{n} \int_{[0,1)^2} \Phi_k(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 2,$$

where we used the fact that $1 + \gamma f_k \geq 0$ in the first as well as item (v) of Remark 2.27 in the last step. This observation establishes the upper bound for the inner product of the discrepancy function with Φ in (2.9).

With a view to the expansion of Φ into a sum of functions Φ_k, $1 \leq k \leq n$, as in (2.14), the proof of the lower bound adheres to the following strategy. First of all, we show that $\langle D_N(\mathcal{P}, \cdot), \Phi_k \rangle$ amounts to at least $\log N$ for $k = 1$, and, secondly, that the contributions for $k \geq 2$ are significantly smaller.

The first of these two tasks largely relies on the choice of signs $\alpha(R)$ within our r-functions (see 2.12). The argument we use is also applicable in $d = 3$ and, hence, we state the corresponding lemma below in a more general form than necessary at the moment. See also [8, Lemma 6].

Lemma 2.28. *Let $\mathcal{P}$ be an N-point set in $[0,1)^d$ and let n be chosen such that $2^{n-2} \leq N < 2^{n-1}$. Then, for every parameter $\vec{r} \in \mathbb{H}_n^d$ there exists a choice of signs $\alpha(R)$ in (2.12) such that*

$$\langle D_N(\mathcal{P}, \cdot), f_{\vec{r}} \rangle \geq 2^{-3-2d}.$$

Proof. We put

$$\alpha(R) = \begin{cases} -1 & \text{if } R \cap \mathcal{P} = \emptyset, \\ \operatorname{sgn}\left(\langle D_N(\mathcal{P}, \cdot), h_R \rangle\right) & \text{if } R \cap \mathcal{P} \neq \emptyset. \end{cases} \tag{2.15}$$

Hence,

$$f_{\vec{r}} = - \sum_{\substack{R \in \mathcal{D}_{\vec{r}} \\ R \cap \mathcal{P} = \emptyset}} h_R + \sum_{\substack{R \in \mathcal{D}_{\vec{r}} \\ R \cap \mathcal{P} \neq \emptyset}} \operatorname{sgn}\left(\langle D_N(\mathcal{P}, \cdot), h_R \rangle\right) h_R$$

and, consequently,

$$\langle D_N(\mathcal{P},\cdot), f_{\vec{r}}\rangle = -\sum_{\substack{R\in\mathcal{D}_{\vec{r}}\\ R\cap\mathcal{P}=\emptyset}} \langle D_N(\mathcal{P},\cdot), h_R\rangle + \sum_{\substack{R\in\mathcal{D}_{\vec{r}}\\ R\cap\mathcal{P}\neq\emptyset}} |\langle D_N(\mathcal{P},\cdot), h_R\rangle|$$
$$\geq -\sum_{\substack{R\in\mathcal{D}_{\vec{r}}\\ R\cap\mathcal{P}=\emptyset}} \langle D_N(\mathcal{P},\cdot), h_R\rangle\,. \quad (2.16)$$

Furthermore, we observe that the counting part of the discrepancy function remains constant on *empty* rectangles, i.e. on dyadic rectangles R with $R\cap\mathcal{P}=\emptyset$. This immediately implies

$$\langle \mathcal{A}(\mathcal{P},N,\cdot), h_R\rangle = \int_{[0,1)^d} \mathcal{A}(\mathcal{P},N,\boldsymbol{x})h_R(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = c\int_R h_R(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0$$

for all empty R and where $c\geq 0$ denotes some constant. It is easy to check that the linear part of the discrepancy function is subject to

$$\int_{[0,1)^d} N x_1 x_2\cdots x_d h_R(x_1,x_2,\ldots,x_d)\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\,\mathrm{d}x_d = N\cdot\frac{|R|^2}{4^d}. \quad (2.17)$$

Additionally, our choice of n guarantees that the number of empty rectangles is at least 2^{n-1}. Indeed, the total number of dyadic rectangles within $\mathcal{D}_{\vec{r}}$ is 2^n (recall that $\|\vec{r}\|_{\ell^1}=n$) and the number of non-empty rectangles is bounded by the number of points N. Additionally, considering $N<2^{n-1}$ verifies the claim.

Continuing with (2.16) we thus obtain

$$\langle D_N(\mathcal{P},\cdot), f_{\vec{r}}\rangle \geq \sum_{\substack{R\in\mathcal{D}_{\vec{r}}\\ R\cap\mathcal{P}=\emptyset}} N\cdot\frac{|R|^2}{4^d} \geq 2^{n-1}2^{n-2}\frac{2^{-2n}}{4^d} = 2^{-3-2d}.$$

□

For the contribution of the functions Φ_k, $k\geq 2$, we rely on the following lemma below, which can also be found in [8, Lemma 17].

Lemma 2.29. *For all N-point sets $\mathcal{P}\subseteq[0,1)^d$ and every r-function with parameter $\vec{s}\in\mathbb{H}^d_s$ there exists a constant $c_d>0$ such that*

$$|\langle D_N(\mathcal{P},\cdot), f_{\vec{s}}\rangle| \leq c_d N 2^{-s}.$$

Proof. From (2.17) it immediately follows that

$$\int_{[0,1)^d} N x_1 x_2 \cdots x_d f_{\vec{s}}(x_1, x_2, \ldots, x_d)\, \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_d = 2^s N \cdot \frac{2^{-2s}}{4^d} = N 2^{-s-2d}.$$

In order to estimate the inner product of $f_{\vec{s}}$ with the counting part we first of all notice that that

$$\mathcal{A}(\mathcal{P}, N, \cdot) = \sum_{\boldsymbol{p} \in \mathcal{P}} \mathbb{1}_{[\boldsymbol{p}, \mathbf{1})},$$

where we define $[\boldsymbol{p}, \mathbf{1}) = [p_1, 1) \times [p_2, 1) \times \cdots \times [p_d, 1)$ for $\boldsymbol{p} = (p_1, p_2, \ldots, p_d) \in \mathcal{P}$.

Due to the geometry of dyadic rectangles each such point $\boldsymbol{p}$ lies in exactly one $R' \in \mathcal{D}_{\vec{s}}$. Similarly to the proof of Lemma 2.28 we see that h_R is orthogonal to $\mathbb{1}_{[\boldsymbol{p}, \mathbf{1})}$ for every $R \in \mathcal{D}_{\vec{s}}$, $R \neq R'$. Therefore,

$$\left| \left\langle \mathbb{1}_{[\boldsymbol{p}, \mathbf{1})}, f_{\vec{s}} \right\rangle \right| = \left| \left\langle \mathbb{1}_{[\boldsymbol{p}, \mathbf{1})}, h_{R'} \right\rangle \right| \leq |R'| = 2^{-s}$$

and, hence,

$$|\langle \mathcal{A}(\mathcal{P}, N, \cdot), f_{\vec{s}} \rangle| \leq N 2^{-s}.$$

□

As a direct result of Lemma 2.28 together with $n \geq \log_2 N + 1$ we obtain

$$\langle D_N(\mathcal{P}, \cdot), \Phi \rangle \geq 3C \cdot \log N - \sum_{k=2}^{n} |\langle D_N(\mathcal{P}, \cdot), \Phi_k \rangle|, \qquad C > 0. \tag{2.18}$$

In order to apply Lemma 2.29 we take a closer look at the functions Φ_k as defined in (2.14) and adhere to last paragraphs of [8, Section 2.4.5]. In dimension $d = 2$ we learn from Remark 2.27, items (ii) and (iv), that each of the summands $f_{j_1} f_{j_2} \cdots f_{j_k}$ is an r-function with parameter $\vec{s} = (n - j_1, j_k)$. We abbreviate $s = \|\vec{s}\|_{\ell^1} = n - j_1 + j_k$ and aim for rearranging the above sum over k in such a way that we obtain an outter sum over s. Evidently, $n + 1 \leq s \leq 2n$. Furthermore, there are at most $2n - s + 1$ pairs (j_1, j_k) which yield a fixed value for s. Indeed, by definition we have $n \geq j_k = j_1 + s - n$, which is possible for $j_1 \in \{0, 1, \ldots, 2n - s\}$. Once j_1 and j_k are fixed, the remaining parameters $j_2, \ldots, j_{k-1}$ can be chosen in $\binom{j_k - j_1 - 1}{k-2} = \binom{s-n-1}{k-2}$ ways. Hence, we may also confine ourselves to values $2 \leq k \leq s - n + 1$. All in all we thus obtain

$$\begin{aligned}
\sum_{k=2}^{n} |\langle D_N(\mathcal{P},\cdot),\Phi_k\rangle| &= \sum_{s=n+1}^{2n}(2n-s+1)\sum_{k=2}^{s-n+1}\binom{s-n-1}{k-2}|\langle D_N(\mathcal{P},\cdot),\Phi_k\rangle| \\
&\le \sum_{s=n+1}^{2n}(2n-s+1)\sum_{k=2}^{s-n+1}\binom{s-n-1}{k-2}\gamma^k c_2 N 2^{-s} \\
&\le c_2\gamma^2 N 2^{-s}\sum_{s=n+1}^{2n}(2n-s+1)\sum_{k=0}^{s-n-1}\binom{s-n-1}{k}\gamma^k \\
&\le \frac{c_2\gamma^2}{4}2^{n-s+1}\sum_{s=n+1}^{2n} n\,(1+\gamma)^{s-n-1} \le \frac{c_2\gamma^2}{4}n\sum_{s=n+1}^{\infty}\left(\frac{1+\gamma}{2}\right)^{s-n-1} \\
&= \frac{c_2\gamma^2}{2(1-\gamma)}n.
\end{aligned}$$

Here, we used Lemma 2.29 in the second, $N < 2^{n-1}$ in the fourth and $0 < \gamma < 1$ in the last step. With a view to (2.18) we recall that, due to our choice of n, we have $n \le \log_2 N + 2$ and we may choose γ sufficiently small such that the last expression in the inequality above can be upper-bounded by $C \log N$. Consequently, we complete the proof of (2.8) by noticing that

$$\langle D_N(\mathcal{P},\cdot),\Phi\rangle \ge 3C\log N - C\log N = 2C\log N, \qquad C > 0.$$

To conclude this section we once again want to point out that for the estimation of the inner product from both above and below it is essential that the products of r-functions involved in the Riesz product Φ yield an r-function again. Equivalently, this can be formulated as "the corresponding parameters need to be strongly distinct". While this is automatically the case in two dimensions (fixing one entry of a hyperbolic vector automatically determines the other coordinate), this is obviously not true in dimensions $d \ge 3$. Hence, we need to work around this subject. One tool that efficiently allows us to do so is presented in the following Section 2.3.2 below.

2.3.2 The Littlewood–Paley inequalities

As it has been mentioned on several occasions in this thesis already, we require a tool to work around several orthogonality issues within our arguments in order to make the machinery of Halász work in $d = 3$. Hence, at some point we need to take care of instances where certain hyperbolic vectors are not strongly distinct. One way to achieve this, is to find a natural substitute for Parseval's identity in L^p-spaces, $1 < p < \infty$, which, at the same time, allows to handle the combinatorial issues arising from the vast complexity of the possible occurences of coincidences. This natural substitute are the so-called (dyadic) Littlewood–Paley inequalities. It needs to be added that, despite their relatively recent introduction to the field of discrepancy theory, their application has lead to numerous remarkable new results, see [11, 13, 15, 16, 29, 30, 44, 51–53] and some recently developed techniques are well explained in [17].

Before we continue we shall introduce some notation. In all that follows we write $A \lesssim B$ if there exists an absolute constant $c > 0$ such that $A \leq cB$. Furthermore, we shall abbreviate $A \lesssim B \lesssim A$ as $A \simeq B$. The implied constant c may depend on all occuring parameters except for the size of the point set N. It needs to be added that several arguments involve L^p-estimates, where, at a later point in Section 2.3.6, the integrability index p is put into relation with N. Hence, this number must not go into the implied constant either.

Furthermore, in an attempt to avoid tedious fiddling with integrals we switch to a probabilistic nomenclature. For an integrable function $f : [0, 1)^3 \to \mathbb{R}$ we write

$$\mathbb{E}f = \int_{[0,1)^3} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}.$$

Furthermore, given some sets $A, B \subseteq \mathbb{R}^3$ we define

$$\mathbb{P}(A) = \mathbb{E}\mathbb{1}_A \quad \text{as well as} \quad \mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}.$$

Since, at a later stage in the proof, we also want to make use of conditional expectation arguments we take a sigma field $\mathcal{F}$ generated by a finite collection of atoms $\mathcal{F}_{\text{atoms}}$ and define

$$\mathbb{E}(f|\mathcal{F}) = \sum_{A \in \mathcal{F}_{\text{atoms}}} \frac{\mathbb{E}(f\mathbb{1}_A)}{\mathbb{P}(A)} \mathbb{1}_A.$$

Apart from stating the main results we also carry out an application of these inequalites in dimensions two and three, as the arguments used therein are characteristic to our needs and implicitly occur at other points in the proof. This survey minutely follows [8, 11].

We begin our discussion with the one-dimensional case. Let $f \in L^p$ and define the *dyadic square function* of f as

$$\mathscr{S}f = \left[|\mathbb{E}f|^2 + \sum_{k=0}^{\infty} \left(\sum_{J \in \mathcal{D}, |J| = 2^{-k}} \frac{\langle f, h_J \rangle}{|J|} h_J \right)^2 \right]^{\frac{1}{2}}.$$

If we choose f to be of the form $f = \sum_{J\in\mathcal{D}} \alpha(J)h_J$, where the coefficients $\alpha(J)$ are in $\mathbb{R}$ for now, this simplifies to

$$\begin{aligned}\mathscr{S}f &= \left[\sum_{k=0}^{\infty}\left(\sum_{J\in\mathcal{D},|J|=2^{-k}} \alpha(J)h_J\right)^2\right]^{\frac{1}{2}} = \left[\sum_{k=0}^{\infty}\sum_{J\in\mathcal{D},|J|=2^{-k}} \alpha^2(J)h_J^2\right]^{\frac{1}{2}} \\ &= \left[\sum_{J\in\mathcal{D}} \alpha^2(J)\mathbb{1}_J\right]^{\frac{1}{2}},\end{aligned}$$

where we used (i) from Remark 2.27 and $h_J^2 = \mathbb{1}_J$ in the first step, the fact that all dyadic intervals of the same length form a partition of $[0,1)$ in the second step, and, finally, the argument concerning squares of Haar functions again in the last step.

The Littlewood–Paley inequalities in one dimension now read as follows (cp. [81]): For all $1 < p < \infty$ there exist positive constants $A_p \le B_p$ with $A_p \simeq 1 + 1/\sqrt{p-1}$ and (for $p \ge 2$) $B_p \lesssim 1 + \sqrt{p}$ such that

$$A_p\|\mathscr{S}f\|_p \le \|f\|_p \le B_p\|\mathscr{S}f\|_p$$

for all $f \in L^p$.

In the case $d = 2$ these inequalities can be used in the following way. Let $f = \sum_{|R|=2^{-n}} \alpha(R)h_R$, $\alpha(R) \in \{-1, +1\}$. We fix the second coordinate and compute the square function with respect to the first one. In a similar fashion as above we obtain

$$\mathscr{S}f = \left[\sum_{r_1=1}^{n}\left(\sum_{|R|=2^{-n},|J_1|=2^{-r_1}} \alpha(R)h_R\right)^2\right]^{\frac{1}{2}}, \tag{2.19}$$

where the innermost sum as is taken over all $R = (J_1, J_2) \in \mathcal{D}^2$ with the stated restrictions. Observe that, once the length of J_1 is fixed, all dyadic rectangles occuring in the innermost sum are disjoint due to the hyperbolic assumption $|R| = 2^{-n}$. Since, additionally, $\alpha^2(R) = 1$ we obtain

$$\mathscr{S}f = \left[\sum_{r_1=1}^{n}\sum_{|R|=2^{-n},|J_1|=2^{-r_1}} \mathbb{1}_R\right]^{\frac{1}{2}} = \left(\#\mathbb{H}_n^2\right)^{\frac{1}{2}} \simeq n^{\frac{1}{2}}.$$

Here, we used the fact that every point in $[0,1)^2$ is contained in exactly $\#\mathbb{H}_n^2$ dyadic rectangles R. Applying the one-dimensional Littlewood–Paley inequality we thus obtain $\|f\|_p \le B_p\|\mathscr{S}f\|_p \lesssim (pn)^{1/2}$, for $p \ge 2$.

Let now $d = 3$ and $f = \sum_{|R|=2^{-n}} \alpha(R) h_R$, $\alpha^2(R) = 1$. Without any difficulty we obtain (2.19). The subsequent step, however, cannot be repeated in this case, as the rectangles in the innermost sum are not disjoint. To this end we require an alternate version of the Littlewood–Paley inequalities, see [22], [11, Theorem 4.1].

Proposition 2.30 (Littlewood–Paley inequalities)**.** Let $\mathscr{H}$ be a Hilbert space and let $f \in L^p_{\mathscr{H}}([0,1))$, $1 < p < \infty$. That is, $f : [0,1) \to \mathscr{H}$ with $\mathbb{E}|f|^p_{\mathscr{H}} < \infty$, where $|\cdot|_{\mathscr{H}}$ denotes the Hilbert space norm. In analogy to the one-dimensional setting we define

$$\mathscr{S}f = \left[|\mathbb{E}f|^2_{\mathscr{H}} + \sum_{J \in \mathcal{D}} \frac{|\langle f, h_J \rangle|^2_{\mathscr{H}}}{|J|^2} \mathbb{1}_J \right]^{\frac{1}{2}},$$

where $\mathbb{E}f$ as well as $\langle f, h_J \rangle$ are to be understood as Bochner integrals (and hence taking the $\mathscr{H}$-norm makes sense). In this notation the Littlewood–Paley inequalites from above continue to hold with the same asymptotic behaviour of A_p and B_p. I.e., for the sake of completeness,

$$A_p \|\mathscr{S}f\|_p \leq \|f\|_p \leq B_p \|\mathscr{S}f\|_p$$

holds with $B_p \lesssim 1 + \sqrt{p}$ for $p \geq 2$ and $A_p \simeq 1 + \frac{1}{\sqrt{p-1}}$.

We continue with the blocked step from above. That is, on the understanding that each $R \in \mathcal{D}^3$ is written as $R = (J_1, J_2, J_3)$ we obtain

$$\|f\|_p \leq B_p \left\| \left[\sum_{r_1=1}^{n} \left(\sum_{|R|=2^{-n}, |J_1|=2^{-r_1}} \alpha(R) h_R \right)^2 \right]^{\frac{1}{2}} \right\|_p$$

by applying the one-dimensional Littlewood–Paley inequality in the first coordinate as we did in (2.19). Let us introduce a function $F : [0,1) \to \ell^2$ by

$$F(x_2) = \sum_{J_2 \in \mathcal{D}} \left\{ \sum_{|R|=2^{-n}, |J_1|=2^{-r_1}} \alpha(R) \prod_{j \neq 2} h_{J_j}(x_j) \right\}_{r_1=1}^{n} h_{J_2}(x_2).$$

The main observation of this discussion is that, obviously, $\|F(x_2)\|_{\ell^2} = \mathscr{S}f$ with $\mathscr{S}f$ defined as in the one-dimensional case. Thus, an application of the Hilbert space version of the Littlewood–Paley inequality in the second line below yields

$$\|f\|_p \le B_p \|\mathscr{S} f\|_p = B_p \left\| \|F(x_2)\|_{\ell^2} \right\|_p$$
$$\le B_p^2 \left\| \left[\sum_{r_1=1}^{n} \sum_{r_2=1}^{n} \Bigg(\sum_{\substack{|R|=2^{-n} \\ |J_1|=2^{-r_1}, |J_2|=2^{-r_2}}} \alpha(R) h_R \Bigg)^2 \right]^{\frac{1}{2}} \right\|.$$

Similarly as before, the dyadic rectangles in the innermost sum are now disjoint and hence the square can be applied to each summand individually. We proceed as in the two-dimensional case and obtain

$$\|f\|_p \le B_p^2 \left\| \left[\sum_{|R|=2^{-n}} \mathbb{1}_R \right]^{\frac{1}{2}} \right\|_p = B_p^2 \left[\#\mathbb{H}_n^3\right]^{\frac{1}{2}} = B_p^2 \binom{n+2}{2}^{\frac{1}{2}} \lesssim pn.$$

2.3.3 The Riesz product in $d = 3$ and proof of Theorem 2.23

The proof of Theorem 2.23 follows the same spirit as the one by Halász we presented in Section 2.3.1. I.e., in all brevity, we define an auxiliary function Ψ^{sd}, whose L^1-norm is bounded by a constant and whose inner product with the discrepancy function is bounded from below by roughly $(\log N)^{1+\eta}$. The arguments behind this machinery, which were developed in [11], however, are considerably more difficult.

Let us fix an arbitrary N-point set $\mathcal{P} \subseteq [0,1)^3$ and choose n such that $2^{n-2} \le N < 2^{n-1}$, or, in other words, $n \simeq \log N$. Furthermore, we take a small constant $a > 0$, fix $\varepsilon > 0$, $0 < b < 1/4$ and introduce the additional parameters

$$q = an^{\varepsilon}, \qquad \rho = q^{1/2} n^{-1}, \qquad \tilde{\rho} = a q^b n^{-1} = a q^{b-1/2} \rho. \tag{2.20}$$

As a matter of fact, q is defined to be the integral part of an^{ε}. As the fractional part of q is of negligible size, however, we continue to work with q as if it were an integer.

The idea to *shorten* the Riesz product from n factors to some $\tilde{q}$ goes back to Beck [4] and is motivated by certain combinatorial issues. In [11] Bilyk and Lacey traded this combinatorial control for the possibility to use a much larger amount of factors $q > \tilde{q}$. This, however, leads to the necessity for involved analytic tools to control the Riesz product, above all the Littlewood–Paley inequalities from the previous sections, exponential Orlicz spaces as well as certain conditional expectational arguments, which we will encounter in Sections 2.3.4–2.3.6.

The parameter ε hereby serves as our control parameter and is directly determined by how well we can handle combinatorial and analytic issues arising from the Riesz product. In the end, the sought value η from the claim of Theorem 2.23 is given by εb and is largely due to the Beck gain for long coincidences, i.e. the main result of Section 2.3.6. The strategy of the remaining part of Section 2.3 is thus to follow the lines of the proof of Bilyk's and Lacey's paper [11] very closely, while we meticulously keep trace of ε, and to improve upon certain steps in Section 2.3.6 to keep the value for ε as large as possible. The last of these steps is also the main content of [69].

It needs to be mentioned that only one step of the proof, more precisely the last step in the proof of (2.26) from Lemma 2.31, requires $b < 1/4$, which finally leads to the choice of b arbitrarily close to $1/4$ to obtain η in Theorem 2.23. In order to set out the dependence of our arguments on b more clearly, we keep b as a (relatively) free parameter in all our other computations.

With a view to constructing a pertinent Riesz product, we partition the set $[\![n]\!]$ into q parts of, in principle, the same cardinality n/q. We confine ourselves to the simplest such partition, i.e. for $1 \leq v \leq q$ we define

$$I_v = \left\{ \frac{(v-1)n}{q} + 1, \frac{(v-1)n}{q} + 2, \ldots, \frac{vn}{q} \right\}. \tag{2.21}$$

Moreover, on the basis of these sets we group hyperbolic vectors into collections $\mathbb{A}_v$, $1 \leq v \leq q$, according to their first coordinate

$$\mathbb{A}_v = \left\{ \vec{r} = (r_1, r_2, r_3) \in \mathbb{H}_n^3 : \ r_1 \in I_v \right\}.$$

Our main building blocks are now given by sums of r-functions whose parameters belong to the same class $\mathbb{A}_v$, $1 \leq v \leq q$,

$$F_v := \sum_{\vec{r} \in \mathbb{A}_v} f_{\vec{r}}.$$

The Riesz product we intend to utilize is now defined as

$$\Psi = \prod_{v=1}^{q} \left(1 + \tilde{\rho} F_v\right). \tag{2.22}$$

Notice that Ψ admits of a useful decomposition similar to the one we had in Halász' proof (2.14), which can be obtained in two steps. First, we simply expand the above product, giving

$$\Psi = 1 + \tilde{\rho}\Psi_1 + \tilde{\rho}^2\Psi_2 + \cdots + \tilde{\rho}^q \Psi_q, \qquad \Psi_v = \sum_{1 \leq j_1 < j_2 < \ldots < j_v \leq q} F_{j_1} F_{j_2} \cdots F_{j_v}. \tag{2.23}$$

For the second step we recall (iv) from Remark 2.27, stating that the product of r-functions is again an r-function if their parameters are strongly distinct. This is incentive enough to split Ψ into

$$\Psi = 1 + \Psi^{\mathrm{sd}} + \Psi^{\neg}, \tag{2.24}$$

where Ψ^{sd} comprises the strongly distinct products of r-functions and $\Psi^{\neg}$ contains the rest. Combining these two expressions we get

$$\Psi^{\mathrm{sd}} = \sum_{v=1}^{q} \Psi_v^{\mathrm{sd}}, \qquad \Psi_v^{\mathrm{sd}} = \tilde{\rho}^v \sum_{1 \le j_1 < \ldots < j_v \le q} \sum_{\substack{(\vec{r}_1,\ldots,\vec{r}_v) \in \mathbb{A}_{j_1} \times \cdots \times \mathbb{A}_{j_v} \\ \vec{r}_1,\ldots,\vec{r}_v \text{ strongly distinct}}} f_{\vec{r}_1} \cdots f_{\vec{r}_v}. \tag{2.25}$$

Similarly one can obtain a formula for $\Psi^{\neg}$. We do not state it here, however, as we will use an alternative representation which occurs in Section 2.3.6. The final choice for the coefficients $\alpha(R)$ hidden within the r-functions is the same as we used in Lemma 2.28 and will thus only become essential in Section 2.3.7.

The central lemma we intend to proof is the following.

Lemma 2.31 (Main lemma, cf. [11, Lemma 7.8])**.** *One has the following estimates*

$$\|\Psi\|_1 \lesssim 1, \tag{2.26}$$
$$\|\Psi^{\neg}\|_1 \lesssim 1, \tag{2.27}$$
$$\|\Psi^{sd}\|_1 \lesssim 1, \tag{2.28}$$

where we require $0 < b < 1/4$ *and* $\varepsilon < \min\{1/3, 1/(1+12b)\}$ *for (2.26) and* $\varepsilon < \varepsilon^{\tau}(b)$, $0.18\ldots \le b < 1/2$ *for (2.27) and (2.28), respectively, where*

$$\varepsilon^{\tau}(b) = \frac{4}{25 + 28b + \sqrt{(3+4b)(155+36b)}}.$$

The proofs of (2.26) and (2.27) are rather involved and are demonstrated in Section 2.3.5 and Section 2.3.6, respectively. Subsequently, (2.28) is an immediate consequence of the expansion (2.24), the triangle inequality and the other two estimates of this lemma.

Proof of Theorem 2.23. We choose Ψ^{sd} to be our auxiliary function. In complete analogy to (2.8) we use Hölder's inequality and (2.25) of Lemma 2.31 to find

$$\left\langle D_N(\mathcal{P},\cdot), \Psi^{\mathrm{sd}} \right\rangle \lesssim D_N^*(\mathcal{P})$$

for all $\varepsilon < \varepsilon^\tau(b)$.

In the other direction we again proceed in the same spirit as in Halász' approach. We choose the r-functions involved as in Lemma 2.28 to find that

$$\left\langle D_N(\mathcal{P},\cdot),\Psi_1^{\mathrm{sd}}\right\rangle = \sum_{v=1}^{q}\sum_{\vec{r}\in\mathbb{A}_v}\tilde{\rho}\left\langle D_N(\mathcal{P},\cdot),f_{\vec{r}}\right\rangle \gtrsim \tilde{\rho}\sum_{v=1}^{q}\sum_{\vec{r}\in\mathbb{A}_v}1\simeq q^b n.$$

In order to estimate the higher order terms Ψ_v^{sd} , we have to deal with further combinatorial issues in Section 2.3.7. That being said, we invoke Lemma 2.48 and arrive at

$$\left\langle D_N(\mathcal{P},\cdot),\Psi^{\mathrm{sd}}\right\rangle \geq \left\langle D_N(\mathcal{P},\cdot),\Psi_1^{\mathrm{sd}}\right\rangle - \sum_{v=2}^{q}\left|\left\langle D_N(\mathcal{P},\cdot),\Psi_v^{\mathrm{sd}}\right\rangle\right| \gtrsim aq^b n. \quad (2.29)$$

Additionally, observe that $b\varepsilon^\tau(b)$ is increasing in b. Since $b < 1/4$ and $\varepsilon^\tau(1/4)/4 - \epsilon = \eta$, we have that n^η is the gain over previous estimates.

□

2.3.4 The Beck gain for simple coincidences

The term *Beck gain* originates from the paper [11]. According to its authors this name was chosen, as Beck discovered in [4] that the L^2-norm of sums of products of not strongly distinct r-functions is smaller than expected. At first, we investigate this phenomenon in the case where only two vectors are involved in the coincidence. The other case is dealt with in Section 2.3.6.

In what follows we use an abbreviation for sums of products of hyperbolic r-functions. That is, for $\mathbb{B} \subseteq (\mathbb{H}_n^3)^k$ we write

$$\mathrm{SP}(\mathbb{B}) = \sum_{(\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_k)\in\mathbb{B}} f_{\vec{r}_1}f_{\vec{r}_2}\cdots f_{\vec{r}_k}.$$

Most of our effort is put into the study of the sets

$$\mathbb{X}_1 = \{(\vec{r},\vec{s})\in\mathbb{A}_t^2 : \vec{r}\neq\vec{s}, r_1 = s_1\},\ \mathbb{X}_2 = \{(\vec{r},\vec{s})\in\mathbb{A}_t\times\mathbb{A}_u : \vec{r}\neq\vec{s}, r_2 = s_2\}.$$

Lemma 2.32 (Beck gain for simple coincidences, cf. [11, Lemma 8.2])**.** *For all $\varepsilon < 1/2$ and all $2 \leq p \leq (n/q)^{1/2}$ we have*

$$\|\mathrm{SP}(\mathbb{X}_1)\|_p \lesssim c_{p,q} n^{\frac{3}{2}}, \qquad c_{p,q} = \max\left\{pq^{-\frac{1}{2}}, p^{\frac{3}{2}}q^{-1}\right\}$$

and for $p \leq q^{3/2}n$

$$\|\mathrm{SP}(\mathbb{X}_2)\|_p \lesssim p^{\frac{3}{2}}q^{-\frac{1}{4}}n^{\frac{3}{2}}.$$

It hardly comes as a surprise that many arguments are more or less the same for both $\mathbb{X}_1$ and $\mathbb{X}_2$. So we implement the proof for $\mathbb{X}_1$ and state both results whenever they differ and provide additional reasoning wherever it is necessary. The proof of this lemma largely relies on the Littlewood–Paley inequalites as well as combinatorial arguments arising from the hyperbolic assumption and is included at the end of this section. The main difference in dealing with the above sets concerns the latter aspect. Consider $\vec{r}$ of some tuple $(\vec{r}, \vec{s})$, for instance. Its first coordinate lies in some $\mathbb{A}_t$ and, hence, it allows for only n/q free choices instead of n. Thus, it *does* make a difference whether we are in the case $r_1 = s_1$ or $r_2 = s_2$.

In the following, we investigate certain subsets of $\mathbb{X}_1^2$ (and $\mathbb{X}_2^2$). The first one, we denote it by $\mathbb{Y}_1$ (or $\mathbb{Y}_2$), comprises mutually distinct vectors $(\vec{r}, \vec{s}, \vec{t}, \vec{u}) \in \mathbb{X}_1^2$ (or $\mathbb{X}_2^2$), whose maximum in the second and third (or the first and third) coordinate is not unique.

We choose two integers $1 \leq \mu, \nu \leq n$ and specialize further. The set $\mathbb{Y}_1(\mu, \nu)$ consists of all quadruples $(\vec{r}, \vec{s}, \vec{t}, \vec{u}) \in \mathbb{Y}_1$ with $r_1 = s_1 = \mu$ and $t_1 = u_1 = \nu$ whose maximum in the second and third coordinate is attained in $s_2 = u_2$ and in $t_3 = r_3$, respectively. The set $\mathbb{Y}_2$ is defined similarly, with the only differences that μ and ν are found in the second coordinate and that the maximum in the first coordinate appears in $\vec{s}$ and $\vec{u}$. For a better understanding an example for elements of both $\mathbb{Y}_1(\mu, \nu)$ and $\mathbb{Y}_2(\mu, \nu)$ is depicted in Figure 2.10.

$$\begin{array}{ccccccc} \vec{r} & & \vec{s} & \vec{t} & & \vec{u} \\ \mu & = & \mu & \nu & = & \nu \\ r_2 & < & s_2 & t_2 & < & s_2 \\ r_3 & > & s_3 & r_3 & > & u_3 \end{array} \qquad \begin{array}{cccccc} \vec{r} & & \vec{s} & \vec{t} & & \vec{u} \\ r_1 & < & s_1 & t_1 & < & s_1 \\ \mu & = & \mu & \nu & = & \nu \\ r_3 & > & s_3 & r_3 & > & u_3 \end{array}$$

Figure 2.10: Elements of $\mathbb{Y}_1(\mu, \nu)$ (left) and $\mathbb{Y}_2(\mu, \nu)$ (right).

Lemma 2.33 (Cf. [11, p. 100f.]). *Let $1 \leq \mu, \nu \leq n$. Then we have the following estimates*

$$\| \mathrm{SP}(\mathbb{Y}_1(\mu, \nu)) \|_p \lesssim pn \qquad \text{and} \qquad \| \mathrm{SP}(\mathbb{Y}_2(\mu, \nu)) \|_p \lesssim pq^{-\frac{1}{2}} n.$$

Proof. We fix μ and ν as above, w.l.o.g. $\mu > \nu$, and define three further sets in each case. That is, firstly,

$$\begin{aligned} \mathbb{X}_1^{(1)}(\mu, \nu) &= \{(\vec{r}, \vec{t}) \in \mathbb{A}_t^2 \colon\ \vec{r} \neq \vec{t}, r_1 = \mu, t_1 = \nu, r_3 = t_3\}, \\ \mathbb{X}_1^{(2)}(\mu, \nu) &= \{(\vec{s}, \vec{u}) \in \mathbb{A}_u^2 \colon\ \vec{s} \neq \vec{u}, s_1 = \mu, u_1 = \nu, s_2 = u_2\}, \text{ and} \\ \mathbb{Z}_1(\mu, \nu) &= \{(\vec{r}, \vec{s}, \vec{t}, \vec{u}) \colon\ (\vec{r}, \vec{t}) \in \mathbb{X}_1^{(1)}(\mu, \nu), (\vec{s}, \vec{u}) \in \mathbb{X}_1^{(2)}(\mu, \nu) \\ &\qquad \text{and the coincidence in the 2nd or 3rd coordinate is not maximal}\}. \end{aligned}$$

In complete analogy we secondly define

$$\begin{aligned}
\mathbb{X}_2^{(1)}(\mu,\nu) &= \{(\vec{r},\vec{t}) \in \mathbb{A}_t^2\colon\ \vec{r} \neq \vec{t}, r_2=\mu, t_2=\nu, r_3=t_3\},\\
\mathbb{X}_2^{(2)}(\mu,\nu) &= \{(\vec{s},\vec{u}) \in \mathbb{A}_u^2\colon\ \vec{s} \neq \vec{u}, s_2=\mu, u_2=\nu, s_1=u_1\}, \text{ and}\\
\mathbb{Z}_2(\mu,\nu) &= \{(\vec{r},\vec{s},\vec{t},\vec{u})\colon\ (\vec{r},\vec{t}) \in \mathbb{X}_2^{(1)}(\mu,\nu), (\vec{s},\vec{u}) \in \mathbb{X}_2^{(2)}(\mu,\nu)\\
&\qquad \text{and the coincidence in the 1st or 3rd coordinate is not maximal}\}.
\end{aligned}$$

This allows us to rewrite $\mathrm{SP}(\mathbb{Y}_j(\mu,\nu))$, $j=1,2$, as

$$\mathrm{SP}\left(\mathbb{Y}_j(\mu,\nu)\right) = \mathrm{SP}\left(\mathbb{X}_j^{(1)}(\mu,\nu)\right)\cdot \mathrm{SP}\left(\mathbb{X}_j^{(2)}(\mu,\nu)\right) - \mathrm{SP}\left(\mathbb{Z}_j(\mu,\nu)\right). \quad (2.30)$$

We estimate each of the three entities on the right-hand side separately. First we show

$$\|\mathrm{SP}(\mathbb{X}_1^{(1)}(\mu,\nu))\|_p \lesssim p^{\frac{1}{2}} n^{\frac{1}{2}}, \qquad \|\mathrm{SP}(\mathbb{X}_2^{(1)}(\mu,\nu))\|_p \lesssim p^{\frac{1}{2}} q^{-\frac{1}{2}} n^{\frac{1}{2}}. \quad (2.31)$$

Due to the hyperbolic assumption we are in the following situation

$$\mathbb{X}_1^{(1)}(\mu,\nu)\colon\ \begin{array}{ccc} \vec{r} & & \vec{t} \\ \mu & > & \nu \\ r_2 & < & t_2 \\ r_3 & = & t_3 \end{array}, \qquad \mathbb{X}_2^{(1)}(\mu,\nu)\colon\ \begin{array}{ccc} \vec{r} & & \vec{t} \\ r_1 & < & t_1 \\ \mu & > & \nu \\ r_3 & = & t_3 \end{array}.$$

We apply the Littlewood–Paley inequality in the second coordinate and obtain

$$\left\|\mathrm{SP}\left(\mathbb{X}_1^{(1)}(\mu,\nu)\right)\right\|_p \lesssim \sqrt{p}\left\|\left[\sum_{t_2=1}^{n}\Big|\sum_{r_2<t_2,\ r_3=t_3} f_{\vec{r}} f_{\vec{t}}\Big|^2\right]^{\frac{1}{2}}\right\|_p.$$

Notice that the values for r_2 and r_3 are already determined once t_2 is fixed. Hence,

$$\left\|\mathrm{SP}\left(\mathbb{X}_1^{(1)}(\mu,\nu)\right)\right\|_p \lesssim \sqrt{p}\left\|\left[\sum_{t_2=1}^{n} f_{\vec{r}}^2 f_{\vec{t}}^2\right]^{\frac{1}{2}}\right\|_p = p^{\frac{1}{2}} n^{\frac{1}{2}}.$$

The estimate for $\mathbb{X}_2^{(1)}(\mu,\nu)$ is derived in the same spirit, only now we have to apply the Littlewood–Paley inequality w.r.t. the first coordinate and hence the outer sum runs through $\mathbb{A}_t$ instead of $[\![n]\!]$. Thus,

$$\left\|\mathrm{SP}\left(\mathbb{X}_2^{(1)}(\mu,\nu)\right)\right\|_p \lesssim \sqrt{p}\left\|\left[\sum_{t_1\in\mathbb{A}_t}\Big|\sum_{r_1<t_1,\ r_3=t_3} f_{\vec{r}} f_{\vec{t}}\Big|^2\right]^{\frac{1}{2}}\right\|_p = p^{\frac{1}{2}} q^{-\frac{1}{2}} n^{\frac{1}{2}}.$$

Following the same strategy as above we get

$$\| \operatorname{SP}(\mathbb{X}_1^{(2)}(\mu,\nu))\|_p \lesssim p^{\frac{1}{2}} n^{\frac{1}{2}}, \tag{2.32}$$

i.e., we apply the Littlewood–Paley inequality once, but now in the third coordinate and, subsequently, everything follows as above. Observe that for $\mathbb{X}_2^{(2)}$ we have to apply the Littlewood–Paley inequality with respect to the third coordinate and therefore we have no additional gain of $q^{-1/2}$. Thus,

$$\| \operatorname{SP}(\mathbb{X}_2^{(2)}(\mu,\nu))\|_p \lesssim p^{\frac{1}{2}} n^{\frac{1}{2}}. \tag{2.33}$$

Finally, let us consider the sets $\mathbb{Z}_j(\mu,\nu)$. We show

$$\|\operatorname{SP}(\mathbb{Z}_1(\mu,\nu))\|_p \lesssim pn \qquad \text{and} \qquad \|\operatorname{SP}(\mathbb{Z}_2(\mu,\nu))\|_p \lesssim pnq^{-\frac{1}{2}}. \tag{2.34}$$

The situation for $\mathbb{Z}_1(\mu,\nu)$ can be depicted as follows

$$\begin{array}{ccccccc} \vec{r} & & \vec{s} & & \vec{t} & & \vec{u} \\ \mu & = & \mu & > & \nu & = & \nu \\ r_2 & & s_2 & & t_2 & & s_2 \\ r_3 & & s_3 & & r_3 & & u_3 \end{array}. \tag{2.35}$$

Let us consider an arbitrary quadrupel $(\vec{r},\vec{s},\vec{t},\vec{u})$ as in (2.35) above and let us denote $m_2 = \max\{r_2, s_2, t_2\}$ and $m_3 = \max\{r_3, s_3, u_3\}$. The crucial observation is that the positions of m_2 and m_3 in the quadrupel are uniquely determined. Besides, once merely the values for m_2 and m_3 are fixed, all the hyperbolic vectors involved are specified. To see this we assume for contradiction that $r_2 = m_2$. Then

$$n = \|\vec{t}\|_{\ell^1} = \nu + t_2 + r_3 < \mu + r_2 + r_3 = \|\vec{r}\|_{\ell^1} = n,$$

which is absurd and hence $m_2 \in \{s_2, t_2\}$. In the first case, i.e. $m_2 = s_2$, we immediately obtain $m_3 = r_3$, as otherwise $\|\vec{r}\|_{\ell^1} < \|\vec{s}\|_{\ell^1}$ or $\|\vec{t}\|_{\ell^1} < \|\vec{u}\|_{\ell^1}$, which is forbidden by definition. Thus, $m_2 = t_2$. Moreover, if the biggest value in the last row were r_3 then $\|\vec{u}\|_{\ell^1} < \|\vec{t}\|_{\ell^1}$, a contradiction. In the same way we can exclude $s_3 = m_3$ too, as otherwise $\|\vec{u}\|_{\ell^1} < \|\vec{s}\|_{\ell^1}$ and, hence, $u_3 = m_3$. We may thus apply the Littlewood–Paley inequality twice (once w.r.t. m_2 and once w.r.t. m_3) to estimate

$$\|\operatorname{SP}(\mathbb{Z}_1(\mu,\nu))\|_p \lesssim p \left\| \left[\sum_{m_2,m_3} \left| \sum_{\substack{t_2=m_2 \\ u_3=m_3}} f_{\vec{r}}\, f_{\vec{s}}\, f_{\vec{t}}\, f_{\vec{u}} \right|^2 \right]^{\frac{1}{2}} \right\|_p = pn.$$

For $\mathbb{Z}_2(\mu,\nu)$ we can proceed in a very similar fashion. We only have to keep in mind that one application of the Littlewood–Paley inequality affects m_1 instead of m_2. Hence, the outermost sum in the above equation contributes n^2/q instead of n^2 and we obtain (2.34).

Using the triangle inequality on (2.30) and subsequently applying the generalized Hölder's inequality yields

$$\begin{aligned}\|\mathrm{SP}\left(\mathbb{Y}_j(\mu,\nu)\right)\|_p &\le \left\|\mathrm{SP}\left(\mathbb{X}_j^{(1)}(\mu,\nu)\right)\mathrm{SP}\left(\mathbb{X}_j^{(2)}(\mu,\nu)\right)\right\|_p + \|\mathrm{SP}\left(\mathbb{Z}_j(\mu,\nu)\right)\|_p \\ &\le \left\|\mathrm{SP}\left(\mathbb{X}_j^{(1)}(\mu,\nu)\right)\right\|_{2p}\left\|\mathrm{SP}\left(\mathbb{X}_j^{(2)}(\mu,\nu)\right)\right\|_{2p} + \|\mathrm{SP}\left(\mathbb{Z}_j(\mu,\nu)\right)\|_p .\end{aligned}$$

Together with (2.32) (for $j=1$) or (2.33) (for $j=2$) as well as (2.31) and (2.34) the result follows. □

Lemma 2.34 (Cf. [11, Lemma 8.6])**.** *Let $\mathbb{Y}_1$ and $\mathbb{Y}_2$ be as introduced in the paragraphs preceding Lemma 2.33. Then we have*

$$\|\mathrm{SP}(\mathbb{Y}_1)\|_p \lesssim pq^{-2}n^3 \qquad \textit{and} \qquad \|\mathrm{SP}(\mathbb{Y}_2)\|_p \lesssim pq^{-\frac{1}{2}}n^3.$$

Proof. We split $\mathbb{Y}_1$ into three types of subsets according to how many vectors of an arbitrary quadruple $(\vec{r},\vec{s},\vec{t},\vec{u})$ are involved in containing the maxima of the second and third coordinate. First, we consider the case where w.l.o.g. $\vec{r}$ and $\vec{t}$ share their second and the third component. Obviously $\vec{r}=\vec{t}$ in this case, due to the hyperbolic assumption. This, however, contradicts the definition of $\mathbb{Y}_1$.

Secondly, assume that three vectors are involved. As a representative we pick the case where $r_2=t_2$ is the maximum in the second and $r_3=u_3$ is maximal in the third coordinate. By definition we have $r_1=s_1$. Hence

$$\|\vec{s}\|_{\ell^1} = r_1+s_2+s_3 < r_1+r_2+r_3 = \|\vec{r}\|_{\ell^1} = n$$

and, thus, this case is impossible too.

The only possibility left is that the whole quadruple is involved in carrying the maxima, say, in $s_2=u_2$ and $r_3=t_3$. Notice that, for fixed values $\mu=r_1$ and $\nu=t_1$, this is exactly $\mathbb{Y}_1(\mu,\nu)$. Thus, two applications of the triangle inequality yield

$$\|\mathrm{SP}(\mathbb{Y}_1)\|_p \le \sum_{\mu,\nu}\|\mathrm{SP}(\mathbb{Y}_1(\mu,\nu))\|_p \lesssim \left(\frac{n}{q}\right)^2 pn = pq^{-2}n^3,$$

where we used Lemma 2.33 in the last but one step.

Except for the last step, the result for $\mathbb{Y}_2$ can be derived in the same spirit. Only at the end we have to consider n^2 choices for μ and ν. □

Let us now turn to the main result of this section.

Proof of Lemma 2.32. We consider the case $p = 2$ first and immediately see that

$$\|\mathrm{SP}(\mathbb{X}_j)\|_2^2 = \mathbb{E}\left[\sum_{(\vec{r},\vec{s})\in\mathbb{X}_j} f_{\vec{r}}\, f_{\vec{s}}\right]^2 = \sum_{(\vec{r},\vec{s},\vec{t},\vec{u})\in\mathbb{X}_j\times\mathbb{X}_j} \mathbb{E} f_{\vec{r}}\, f_{\vec{s}}\, f_{\vec{t}}\, f_{\vec{u}}.$$

Again, the first step is to narrow down the choices of $(\vec{r},\vec{s},\vec{t},\vec{u})$ which are relevant to us. Recall that, due to the product rule (Proposition 2.25), it suffices that the maximum in *one* coordinate of a quadruple is unique to let the corresponding summand vanish. Thus, we need only consider those terms which have a coincidence in each coordinate. If a coincidence in one coordinate extends over three or four vectors then $\vec{r} = \vec{s}$ or $\vec{t} = \vec{u}$. This is impossible as they are supposed to be distinct by definition. Thus, $\mathbb{X}_j \times \mathbb{X}_j$ decays into two sets, $\mathbb{Y}_j$ and $\tilde{\mathbb{Y}}_j$. The first one already occured several times in this section and the latter is defined by

$$\tilde{\mathbb{Y}}_j = \{(\vec{r},\vec{s},\vec{t},\vec{u}) \in \mathbb{X}_j^2 : \text{not all vectors are distinct and the maximum in the } (3-j)\text{-th and 3rd coordinate is not unique}\}.$$

The set $\tilde{\mathbb{Y}}_j$ is easy to handle, as it basically admits of two types of constellations only, namely $(\vec{r},\vec{s},\vec{r},\vec{s})$ and $(\vec{r},\vec{s},\vec{r},\vec{u})$, $\vec{s} \neq \vec{u}$. The second case looks as follows (for $j = 1$):

$$\begin{array}{ccccccc} \vec{r} & & \vec{s} & & \vec{t} & & \vec{u} \\ r_1 & = & r_1 & = & r_1 & = & r_1 \\ r_2 & & s_2 & & r_2 & & u_2 \\ r_3 & & s_3 & & r_3 & & u_3 \end{array}.$$

If the highest value in the second coordinate occurs in r_2, then the maximum in the last row is either attained in s_3 or in u_3. As the maximum is not unique we necessarily have $s_3 = u_3$ and consequently $\vec{s} = \vec{u}$, which is impossible. If the maximum in the second coordinate is s_2 or u_2 we have $s_2 = u_2$ leading to a contradiction as before. The same argument shows that the second constellation $(\vec{r},\vec{s},\vec{r},\vec{u})$ cannot occur for $j = 2$ either. Therefore we have

$$\mathrm{SP}(\tilde{\mathbb{Y}}_j) = \sum_{(\vec{r},\vec{s})\in\mathbb{X}_j} f_{\vec{r}}^2 f_{\vec{s}}^2 = \#\mathbb{X}_j, \tag{2.36}$$

which is $q^{-1}n^3$ if $j = 1$ and $q^{-2}n^3$ if $j = 2$. This finishes the case $p = 2$.

Let us now consider $p \geq 4$. We define

$$N_j(p) = \left\|\mathrm{SP}(\mathbb{X}_j)\right\|_p .$$

As we have already done before, applying the Littlewood–Paley inequality twice yields

$$N_1(p) \lesssim p \left\| \left[\sum_{m_2,m_3} \left| \sum_{\substack{(\vec{r},\vec{s})\in\mathbb{X}_1 \\ \max\{r_2,s_2\}=m_2, \max\{r_3,s_3\}=m_3}} f_{\vec{r}}\, f_{\vec{s}} \right|^2 \right]^{\frac{1}{2}} \right\|_p . \tag{2.37}$$

For $N_2(p)$ we obtain the same expression with the outer summation index m_2 replaced by m_1, $\mathbb{X}_1$ by $\mathbb{X}_2$, and the inner sum going over all $m_1 = \max\{r_1, s_1\}$ instead of $m_2 = \max\{r_2, s_2\}$. The expression under the square root can be rearranged into three main parts. First, the diagonal elements, secondly, the terms where two vectors are equal, and, thirdly, those terms where none of the vectors are equal. This leads to

$$\sum_{m_2,m_3} \left| \sum_{\substack{(\vec{r},\vec{s})\in\mathbb{X}_1 \\ \max\{r_2,s_2\}=m_2, \max\{r_3,s_3\}=m_3}} f_{\vec{r}} f_{\vec{s}} \right|^2 \simeq \mathrm{SP}\left(\tilde{\mathbb{Y}}_1\right) + \sum_{1\leq k<l\leq 4} \mathrm{SP}\left(\mathbb{Y}_1^{(k,l)}\right) + \mathrm{SP}(\mathbb{Y}_1),$$

where

$$\mathbb{Y}_j^{(k,l)} = \left\{(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4) \in \mathbb{X}_j^2 :\ \vec{r}_k = \vec{r}_l \text{ and the other two are distinct}\right\}$$

for $j = 1, 2$, and where $\tilde{\mathbb{Y}}_1$ and $\mathbb{Y}_1$ have already been used earlier in the proof and their norms can be estimated with the help of (2.36) and Lemma 2.34.

For the middle term we observe that $\mathbb{Y}_1^{(1,2)} = \mathbb{Y}_1^{(3,4)} = \emptyset$ by definition of $\mathbb{X}_1$. Furthermore, we have

$$\mathrm{SP}\left(\mathbb{Y}_1^{(1,3)}\right) = \sum_{(\vec{r},\vec{s},\vec{r},\vec{u})\in\mathbb{X}_1^2} f_{\vec{r}}^2\, f_{\vec{s}}\, f_{\vec{u}} = n \sum_{(\vec{s},\vec{u})\in\mathbb{X}_1} f_{\vec{s}}\, f_{\vec{u}} = n\, \mathrm{SP}(\mathbb{X}_1),$$

for instance. The second equality holds since, first of all, if $\vec{s}$ is given we only need one more parameter (r_2 or r_3) to fully specify $\vec{r}$ and, secondly, due to the special structure of the quadruple $(\vec{r}, \vec{s}, \vec{r}, \vec{u})$ we can claim that $(\vec{s}, \vec{u}) \in \mathbb{X}_1$, indeed.

Up to this point, the same line of reasoning can be used for $N_2(p)$ as well, except for the obvious modifications of indices. In the last step, however, the number of free parameters can be reduced to n/q as we may fix r_1 to fully determine $\vec{r}$.

We consider the above discussion in (2.37) and use the subadditivity of the square root as well as the triangle inequality to obtain

$$\begin{aligned} N_j(p) &\lesssim p \left\| \left[\#\mathbb{X}_j + \sum_{1 \le k < l \le 4} \mathrm{SP}\left(\mathbb{Y}_j^{(k,l)}\right) + \mathrm{SP}(\mathbb{Y}_j) \right]^{\frac{1}{2}} \right\|_p \\ &\le p \left[(\#\mathbb{X}_j)^{\frac{1}{2}} + \sum_{1 \le k < l \le 4} \left\| \mathrm{SP}\left(\mathbb{Y}_j^{(k,l)}\right) \right\|_{p/2}^{\frac{1}{2}} + \|\mathrm{SP}(\mathbb{Y}_j)\|_{p/2}^{\frac{1}{2}} \right]. \qquad (2.38) \end{aligned}$$

At this point we need to specialize again. To this end let $j = 1$. We continue with the above estimate

$$\begin{aligned} N_1(p) &\lesssim p \left[q^{-\frac{1}{2}} n^{\frac{3}{2}} + n^{\frac{1}{2}} \|\mathrm{SP}(\mathbb{X}_1)\|_{p/2}^{\frac{1}{2}} + p^{\frac{1}{2}} q^{-1} n^{\frac{3}{2}} \right] \\ &\le \max\{pq^{-\frac{1}{2}}, p^{\frac{3}{2}} q^{-1}\} n^{\frac{3}{2}} + p n^{\frac{1}{2}} \left[N_1(p/2) \right]^{\frac{1}{2}}. \end{aligned}$$

That is, for $p = 2^{v+1}$ we obtain

$$N_1(2^{v+1}) \lesssim \max\{2^{v+1} q^{-\frac{1}{2}}, 2^{\frac{3(v+1)}{2}} q^{-1}\} n^{\frac{3}{2}} + 2^{v+1} n^{\frac{1}{2}} \left[N_1(2^v) \right]^{\frac{1}{2}}. \qquad (2.39)$$

On the basis of the above inequality we show

$$N_1(2^{v+1}) \lesssim \max\{2^{v+1} q^{-\frac{1}{2}}, 2^{\frac{3(v+1)}{2}} q^{-1}\} n^{\frac{3}{2}} \qquad (2.40)$$

for all $v \ge 0$, and $(q^{-1}n)^{1/2} \gtrsim 2^v \simeq p$. Indeed, for $v = 0$ this is the case $p = 2$ which was already dealt with in (2.36) and the subsequent line. For an arbitrary v we induct on (2.39) giving

$$N_1(2^{v+1}) \lesssim \max\{2^{v+1} q^{-\frac{1}{2}}, 2^{\frac{3(v+1)}{2}} q^{-1}\} n^{\frac{3}{2}} + 2^{v+1} \max\{2^{\frac{v}{2}} q^{-\frac{1}{4}}, 2^{\frac{3v}{4}} q^{-\frac{1}{2}}\} n^{\frac{5}{4}}$$

by the induction hypothesis (2.40). Suppose

$$2^{v+1} q^{-\frac{1}{2}} \ge 2^{\frac{3(v+1)}{2}} q^{-1}.$$

Then, clearly, $2^{v/2} q^{-1/4} \gtrsim 2^{3v/4} q^{-1/2}$ and, moreover, $q^{-1/2} n^{3/2} \ge 2^{v/2} q^{-1/4} n^{5/4}$ iff $2^v \lesssim (n/q)^{1/2}$. If, on the other hand, $2^{v+1} q^{-1/2} \le 2^{3(v+1)/2} q^{-1}$ we obtain similarly $2^{v/2} q^{-1/4} \lesssim 2^{3v/4} q^{-1/2}$ as well as $2^{3(v+1)/2} q^{-1} n^{3/2} \ge 2^{v+1} 2^{3v/4} q^{-1/2} n^{5/4}$ iff $n/q^2 \gtrsim 2^v$, which, together with $\varepsilon < 1/2$, implies (2.40) and this, in turn, finishes the proof for N_1.

It remains to estimate $N_2(p)$. We have the following result

$$N_2(2^{v+1}) \lesssim 2^{\frac{3(v+1)}{2}} q^{-\frac{1}{4}} n^{\frac{3}{2}}$$

for all $v \geq 0$ and $p \simeq 2^v \lesssim q^{3/2}n$. This is shown in complete analogy to the steps taken above. That is, we start with the counterpart of (2.39)

$$N_2(2^{v+1}) \lesssim 2^{\frac{3(v+1)}{2}} q^{-\frac{1}{4}} n^{\frac{3}{2}} + 2^{v+1} q^{-\frac{1}{2}} n^{\frac{1}{2}} \left[N_2(2^v)\right]^{\frac{1}{2}}.$$

Here we implicitly used $pq^{-1} \leq p^{3/2}q^{-1/4}$. Furthermore, by an inductive argument we obtain

$$N_2(2^{v+1}) \lesssim 2^{\frac{3(v+1)}{2}} q^{-\frac{1}{4}} n^{\frac{3}{2}} + 2^{v+1} 2^{\frac{3v}{4}} q^{-\frac{5}{8}} n^{\frac{5}{4}} \lesssim 2^{\frac{3v}{2}} q^{-\frac{1}{4}} n^{\frac{3}{2}}$$

for $2^v \lesssim q^{3/2}n$, as stated above. □

2.3.5 Norm estimates and the upper bound for the inner product

Let us recall the Riesz product from Section 2.3.3, i.e.

$$\Psi = \prod_{v=1}^{q} (1 + \tilde{\rho} F_v), \quad F_v = \sum_{\vec{r} \in \mathbb{A}_v} f_{\vec{r}}.$$

In what follows we show $\|\Psi\|_1 \lesssim 1$ as claimed in (2.26) from our main Lemma 2.31. Again, we closely follow the strategy of Bilyk, Lacey and Vagharshakyan ([8, 11, 13]) and keep track of the relevant constants. We begin by giving a bound for the L^p-norms of the overall building blocks of Ψ.

Lemma 2.35 (Cf. [13, Theorem 6.1])**.** *For all $1 \leq v < q$ we have*

$$\|\rho F_v\|_p \lesssim p^{\frac{1}{2}}, \qquad p \lesssim \min\{(q^{-1}n)^{\frac{1}{2}}, n^{\frac{1}{3}}\}.$$

More specifically, the bound is obtained for $p \lesssim n^{1/3}$ if $\varepsilon < 1/3$.

Proof. Due to the construction of the functions F_v we can apply the Littlewood–Paley inequality, i.e. Proposition 2.30, in the first coordinate to obtain

$$\|\rho F_v\|_p \lesssim p^{\frac{1}{2}} \left\| \left[\sum_{r \in I_v} \left| \rho \sum_{r_1 = r} f_{\vec{r}} \right|^2 \right]^{\frac{1}{2}} \right\|_p = p^{\frac{1}{2}} \left\| \left[\rho^2 \sum_{\vec{r} \in \mathbb{A}_v} f_{\vec{r}}^2 + \rho^2 \sum_{\substack{\vec{r} \neq \vec{s} \in \mathbb{A}_v \\ r_1 = s_1}} f_{\vec{r}} f_{\vec{s}} \right] \right\|_p^{\frac{1}{2}}$$

$$\leq p^{\frac{1}{2}} \left[(\rho^2 \# \mathbb{A}_v)^{\frac{1}{2}} + \rho \left\| \sum_{\substack{\vec{r} \neq \vec{s} \in \mathbb{A}_v \\ r_1 = s_1}} f_{\vec{r}} f_{\vec{s}} \right\|_{p/2}^{\frac{1}{2}} \right]$$

where we used the triangle inequality and the subadditivity of the square root in the last step.

Recall that $\rho = q^{1/2}n^{-1}$ and, hence, $\rho^2 \# \mathbb{A}_v \simeq 1$. Thus, if we additionally apply the simple Beck gain, Lemma 2.32, to the last expression from above we obtain

$$\|\rho F_v\|_p \lesssim p^{\frac{1}{2}} \left(1 + q^{\frac{1}{2}} n^{-1} c_{p,q}^{\frac{1}{2}} n^{\frac{3}{4}}\right) = p^{\frac{1}{2}} \left(1 + \max\{p^{\frac{1}{2}} q^{\frac{1}{4}}, p^{\frac{3}{4}}\} n^{-\frac{1}{4}}\right).$$

The application of the simple Beck gain requires $p \lesssim q^{-1/2}n^{1/2}$. Furthermore, we have $p^{1/2}q^{1/4} < n^{1/4}$ iff $p < (n/q)^{1/2}$, and $p^{3/4} < n^{1/4}$ iff $p < n^{1/3}$ and, thus, the desired result follows. □

Assume for a moment that the result of the above lemma holds for the full range of p. In this case we could make use of the equivalence of norms (cf. [11, Proposition 5.2])

$$\|f\|_{\exp(L^\alpha)} \simeq \sup_{p\geq 1} p^{-\frac{1}{\alpha}} \|f\|_p \simeq \sup_{\lambda>0} \lambda^\alpha \left|\log \mathbb{P}\left(|f| > \lambda\right)\right|$$

to find that that ρF_v lies in the exponential Orlicz class $\exp(L^2)$ and that it satisfies the distributional estimate

$$\mathbb{P}\left(|\rho F_v| > \lambda\right) \lesssim \exp\left(-c\lambda^2\right), \qquad c > 0,$$

for all $\lambda > 0$. Nevertheless, the proof of the above estimate can be adapted to *moderate values of* λ, according to [8, p. 141]. That is, in [13, p. 2489f] it is stated that one has the inequality

$$\mathbb{P}\left(|\rho F_v| > x\right) \lesssim C^p p^{\frac{p}{2}} x^{-p}$$

valid for some constant C, all $x > 0$, all $1 \leq v \leq q$, and all p meeting the requirements of Lemma 2.35, i.e. $1 \leq p \lesssim n^{1/3}$ for $\varepsilon \leq 1/3$. Choosing $p = x^{2-\delta}$ for an arbitrarily small $\delta > 0$ we further obtain the distributional estimate

$$\mathbb{P}\left(\rho F_v < -x\right) \lesssim \exp(-cx^{2-\delta_0}), \qquad 0 < c < 1, \tag{2.41}$$

for all $x \lesssim n^{1/6}$ and all small $\delta_0 > 0$.

Contrary to Halász' Riesz product $\Phi + 1$, see (2.13), the function Ψ can take on negative values. Nevertheless, the following lemma reflects the fact that this only happens on very small sets.

Lemma 2.36 (Cf. [11, Lemma 7.8]). *For all $\varepsilon < 1/(3(1-2b))$, $b < 1/2$, and $\tilde{\delta} > 0$ we have*

$$\mathbb{P}(\Psi < 0) \lesssim \exp\left(-Aq^{1-2b-\tilde{\delta}}\right),$$

where $A \simeq a^{-2}$.

Proof. Since Ψ is defined as a product (cf. (2.22)) we need to have at least one negative factor for the whole function to be negative. Considering this fact in the second step below thus leads to

$$\mathbb{P}(\Psi < 0) = \mathbb{P}\left(\prod_{v=1}^{q}(1+\tilde{\rho}F_v) < 0\right) \lesssim \sum_{v=1}^{q}\mathbb{P}\left(\tilde{\rho}F_v < -1\right) =$$
$$= \sum_{v=1}^{q}\mathbb{P}\left(\rho F_v < -a^{-1}q^{\frac{1}{2}-b}\right) \lesssim \exp\left(-Ca^{-2}q^{1-2b-\tilde{\delta}}\right),$$

where we used (2.41) and abbreviated $\tilde{\delta} := \delta_0(\frac{1}{2}-b) > 0$ for $b < \frac{1}{2}$. Notice that we require

$$q^{\frac{1}{2}-b} \lesssim n^{\frac{1}{6}}$$

for this step, or, equivalently, $\varepsilon < 1/(3(1-2b))$. □

The lemma below indicates that the L^2-norm of Ψ satisfies an exponentially increasing bound in q. In the course of the proof we apply a interpolating version of Hölder's inequality, which is also sometimes referred to as *Lyapunov's inequality*. I.e., let $f \in L^p$, $1 \le p \le \infty$. Then, for all $\theta \in (0,1)$ and all choices p_0, p_1 such that

$$p = \theta p_0 + (1-\theta)p_1$$

we have

$$\|f\|_p^p \le \|f\|_{p_0}^{\theta p_0} \cdot \|f\|_{p_1}^{(1-\theta)p_1}. \tag{2.42}$$

Lemma 2.37 (Cf. [11, Lemma 7.8])**.** *For all* $\varepsilon < \min\{1/3, 1/(1+12b)\}$ *we have*

$$\|\Psi\|_2 \lesssim \exp\left(a'q^{2b}\right),$$

where $a' \simeq a^2$.

Proof. To facilitate notation let us define

$$G_v := (1+\tilde{\rho}F_v)^2, \qquad 1 \le v \le q.$$

In several steps of the proof we use arguments involving conditional expectations. To this end, let us fix the second and third coordinate and set $\mathcal{F} = \sigma(F_1, F_2, \ldots, F_{q-1})$, i.e., the sigma algebra generated by the first variable of $F_1, F_2, \ldots, F_{q-1}$. Observe that

$$\mathbb{E}(F_q|\mathcal{F}) = \sum_{v=1}^{q-1}\frac{\mathbb{E}(F_q\mathbb{1}_{I_v})}{|I_v|}\mathbb{1}_{I_v} = 0,$$

since F_q is not supported on I_v, $v < q$. Here, I_v denotes a set of the partition as introduced in (2.21). Furthermore, we have

$$\mathbb{E}\left((\tilde{\rho}F_q)^2\middle|\,\mathcal{F}\right) = \tilde{\rho}^2\mathbb{E}\left(\sum_{\vec{r}\in\mathbb{A}_q} f_{\vec{r}}^2\middle|\mathcal{F}\right) + \tilde{\rho}^2\mathbb{E}\left(\sum_{\vec{r}\neq\vec{s}\in\mathbb{A}_q} f_{\vec{r}}f_{\vec{s}}\middle|\mathcal{F}\right).$$

Notice that each summand in the last expression, where $\max\{r_1, s_1\}$ is unique, vanishes due to the last item in Remark 2.27. Thus,

$$\mathbb{E}\left((\tilde{\rho}F_q)^2\middle|\,\mathcal{F}\right) = \tilde{\rho}^2\#\mathbb{A}_q + \tilde{\rho}^2\sum_{\substack{\vec{r}\neq\vec{s}\in\mathbb{A}_q\\ r_1=s_1}} f_{\vec{r}}f_{\vec{s}} = a^2q^{2b-1} + \tilde{\rho}^2\sum_{\substack{\vec{r}\neq\vec{s}\in\mathbb{A}_q\\ r_1=s_1}} f_{\vec{r}}f_{\vec{s}}$$

and, furthermore,

$$\mathbb{E}\left(G_q\middle|\,\mathcal{F}\right) = 1 + a^2q^{2b-1} + \tilde{\rho}^2\sum_{\substack{\vec{r}\neq\vec{s}\in\mathbb{A}_q\\ r_1=s_1}} f_{\vec{r}}f_{\vec{s}}. \tag{2.43}$$

Let us recall a general property of conditional expactations, namely, if Y and Z are random variables and Y is $\mathcal{G}$-measurable, where $\mathcal{G}$ denotes some sigma algebra, then $\mathbb{E}(YZ|\mathcal{G}) = Y\mathbb{E}(Z|\mathcal{G})$. Therefore, we obtain

$$\begin{aligned}\mathbb{E}(G_1G_2\cdots G_q) &= \mathbb{E}\left[\mathbb{E}\left(G_1\cdots G_q|\mathcal{F}\right)\right] = \mathbb{E}[G1\cdots G_{q-1}\mathbb{E}(G_q|\mathcal{F})]\\ &\lesssim \left(1+a^2q^{2b-1}\right)\mathbb{E}[G_1\cdots G_{q-1}] + \mathbb{E}\left[G_1\cdots G_{q-1}\left|\tilde{\rho}^2\sum_{\substack{\vec{r}\neq\vec{s}\in\mathbb{A}_q\\ r_1=s_1}} f_{\vec{r}}f_{\vec{s}}\right|\right]\end{aligned} \tag{2.44}$$

by (2.43).

Let us introduce yet another quantity. For $V \le q$ we define

$$N(V,r) := \left\|\prod_{v=1}^{V}(1+\tilde{\rho}F_v)\right\|_r.$$

Applying the generalized Hölder's inequality in the first and Lemma 2.35 in the last step yields

$$N(V,4) \le \prod_{v=1}^{V}\|1+\tilde{\rho}F_v\|_{4V} \le \prod_{v=1}^{V}\left(1+aq^{b-\frac{1}{2}}\|\rho F_v\|_{4V}\right) \le \left(1+aq^{b-\frac{1}{2}}2V^{\frac{1}{2}}\right)^V.$$

In order to use Lemma 2.35 we require $4V \lesssim n^{1/3}$, i.e. $\varepsilon \le 1/3$ since $V \le q$. Continuing with the above estimate we obtain

$$N(V,4) \le \left(1+q^b\right)^V \lesssim (Cq)^{cq} \tag{2.45}$$

for some constants $C, c > 0$, $c \leq b$.

In what follows, we derive the recursive estimate

$$N(V+1,2) \lesssim N(V,2)\left(1+a^2q^{2b-1}\right)^{\frac{1}{2}}. \tag{2.46}$$

By (2.44) we have

$$\left(N(V+1,2)\right)^2 \lesssim (1+a^2q^{2b-1})\,(N(V,2))^2 + \mathbb{E}\left[G_1G_2\cdots G_V\left|\tilde{\rho}^2\sum_{\substack{\vec{r}\neq\vec{s}\in\mathbb{A}_{V+1}\\ r_1=s_1}} f_{\vec{r}}f_{\vec{s}}\right|\right].$$

We apply Hölder's inequality to the second summand giving

$$\begin{aligned}\mathbb{E}\left[G_1\cdots G_V\left|\tilde{\rho}^2\sum_{\substack{\vec{r}\neq\vec{s}\in\mathbb{A}_{V+1}\\ r_1=s_1}} f_{\vec{r}}f_{\vec{s}}\right|\right] &\leq \left\|\tilde{\rho}^2\sum_{\substack{\vec{r}\neq\vec{s}\in\mathbb{A}_{V+1}\\ r_1=s_1}} f_{\vec{r}}f_{\vec{s}}\right\|_q \mathbb{E}\left[(G_1\cdots G_V)^p\right]^{\frac{1}{p}} \\ &= \left\|\tilde{\rho}^2\sum_{\substack{\vec{r}\neq\vec{s}\in\mathbb{A}_{V+1}\\ r_1=s_1}} f_{\vec{r}}f_{\vec{s}}\right\|_q \mathbb{E}\left[\prod_{v=1}^{V}(1+\tilde{\rho}F_v)^{2p}\right]^{\frac{1}{p}},\end{aligned}$$

$1/p+1/q=1$. Hence,

$$\begin{aligned}(N(V+1,2))^2 \leq{}& \left(1+a^2q^{2b-1}\right)(N(V,2))^2 \\ &+ \left\|\tilde{\rho}^2\sum_{\substack{\vec{r}\neq\vec{s}\in\mathbb{A}_{V+1}\\ r_1=s_1}} f_{\vec{r}}f_{\vec{s}}\right\|_q \left(N\left(V,\frac{2}{1-1/q}\right)\right)^2.\end{aligned}$$

We now make use of Lyapunov's inequality (2.42) with $p_0 = 2$, $p_1 = 4$ and $\theta = 1-1/(q-1)$ to see that

$$N\left(V, 2(1-1/q)^{-1}\right) \leq N(V,2)^{1-\frac{2}{q}}N(V,4)^{\frac{2}{q}}.$$

With the help of (2.45) we can thus estimate further

$$\begin{aligned}(N(V+1,2))^2 \leq{}& \left(1+a^2q^{2b-1}\right)(N(V,2))^2 \\ &+ \left\|\tilde{\rho}^2\sum_{\substack{\vec{r}\neq\vec{s}\in\mathbb{A}_{V+1}\\ r_1=s_1}} f_{\vec{r}}f_{\vec{s}}\right\|_q \left(N\left(V,2\right)\right)^{2(1-\frac{2}{q})}(Cq)^{4c}.\end{aligned}$$

The L^q-norm can be estimated via the Beck gain for simple coincidences (Lemma 2.32) with $p = q$. In particular, this means that

$$\|\mathrm{SP}(\mathbb{X}_1)\|_q \lesssim c_{q,q}n^{\frac{3}{2}} = q^{\frac{1}{2}}n^{\frac{3}{2}}$$

if $q \leq (n/q)^{1/2}$. This is indeed the case for ε as given in the claim. Hence, all the considerations above lead to

$$\begin{aligned}(N(V+1,2))^2 &\lesssim \left(1+a^2q^{2b-1}\right)(N(V,2))^2 + \tilde{\rho}^2 q^{\frac{1}{2}} n^{\frac{3}{2}} (N(V,2))^{2(1-\frac{2}{q})} q^{4c} \\ &\simeq \left(1+a^2q^{2b-1}\right)(N(V,2))^2 + a^2 q^{\frac{1}{2}+2b+4c} n^{-\frac{1}{2}} (N(V,2))^{2(1-\frac{2}{q})}.\end{aligned}$$

To obtain the desired recursive estimate (2.46) two more observations are required. Firstly, let us assume $N(V,2) \geq 1$. Thus, $(N(V,2))^{-4/q} \leq 1$ and the right power of $N(V,2)$ is obtained. We will see at the end of the proof that the case $N(V,2) < 1$ for some V is actually even more benificial to us. In any case, secondly we have

$$q^{\frac{1}{2}+2b+4c} \leq q^{\frac{1}{2}+6b} \leq n^{\frac{1}{2}}$$

for $\varepsilon < 1/(1+12b)$. This completes the proof of (2.46).

Let us now return to our claim. We have

$$\|\Psi\|_2 = N(q,2) \lesssim N(q-1,2)\left(1+a^2q^{2b-1}\right)^{\frac{1}{2}} \lesssim \ldots \lesssim \left(1+a^2q^{2b-1}\right)^{\frac{q}{2}},$$

where we repeatedly applied (2.46). Observe that, if $N(V,2)$ were smaller than 1 for any $V \leq q$, our recursive argument would stop earlier and we would obtain an even better result. Thus, in the worst case, we continue with the above estimate, which finally gives

$$\|\Psi\|_2 \lesssim \left(1+a^2q^{2b-1}\right)^{\frac{q}{2}} \leq \exp\left(a^2q^{2b}/2\right)$$

as $1+x \leq \exp(x)$ for all $x \geq 0$. □

We may now turn to the proof of the first item of Lemma 2.31.

Proof of (2.26). Let us denote by Ψ^+ the positive part and by Ψ^- the negative part of Ψ, i.e.

$$\Psi^+ = \Psi \mathbb{1}_{\{\Psi \geq 0\}} \qquad \text{and} \qquad \Psi^- = \Psi \mathbb{1}_{\{\Psi < 0\}}.$$

In this notation we have $\Psi = \Psi^+ + \Psi^-$ as well as $|\Psi| = \Psi^+ - \Psi^- = \Psi - 2\Psi^-$. Therefore,

$$\begin{aligned}\|\Psi\|_1 = \mathbb{E}\Psi - 2\mathbb{E}\left(\Psi \mathbb{1}_{\{\Psi<0\}}\right) &\leq \mathbb{E}\Psi + 2\left(\mathbb{E}\left(\mathbb{1}_{\{\Psi<0\}}\right)\right)^{\frac{1}{2}} \|\Psi\|_2 \\ &\simeq \mathbb{E}\Psi + (\mathbb{P}(\Psi<0))^{\frac{1}{2}} \|\Psi\|_2,\end{aligned}$$

where we used the Cauchy–Schwarz inequality. It remains to determine the mean of Ψ. We have

$$\mathbb{E}\Psi = 1. \tag{2.47}$$

Indeed, consider the expansion (2.23)

$$\Psi = 1 + \tilde{\rho}\Psi_1 + \tilde{\rho}^2\Psi_2 + \cdots + \tilde{\rho}^q\Psi_q, \qquad \Psi_v = \sum_{1 \le j_1 < j_2 < \ldots < j_v \le q} F_{j_1} F_{j_2} \cdots F_{j_v}.$$

Similarly to Halász' proof, we recall that each F_{j_u} is defined by some $\mathbb{A}_{j_u}$, which in turn is defined by a partition. Hence, for every $1 \le v \le q$, Ψ_v is composed of products of r-functions whose parameters have a unique maximum in the first coordinate. By Remark 2.27 this implies that $\mathbb{E}\Psi_v = 0$ and (2.47) follows.

Considering this as well as Lemma 2.36 and Lemma 2.37 in the above estimate we immediately obtain

$$\|\Psi\|_1 \lesssim 1 + \exp\left(-Aq^{1-2b-\tilde{\delta}} + a'q^{2b}\right) \simeq 1$$

for $a > 0$ sufficiently small and $0 < b \le 1/4 - \tilde{\delta}$. Notice that for b within this range we require

$$\varepsilon < \min\left\{\frac{1}{3}, \frac{1}{1+12b}, \frac{1}{3(1-2b)}\right\} = \min\left\{\frac{1}{3}, \frac{1}{1+12b}\right\}$$

in order to the estimate the probability term as well as the L^2-norm. □

2.3.6 The Beck gain for long coincidences

Here, we emphasize on the not strongly distinct part of Ψ. To be more precise, we proof that the L^1-norm of $\Psi^{\neg}$ is upper-bounded by a constant, i.e. (2.27), and thus conclude the proof of our main Lemma 2.31. Naturally, this involves the study of coincidences within collections of hyperbolic vectors. We have already encountered such problems in a very simple setting in Section 2.3.4. Clearly, coincidences can stretch across a large number of hyperbolic vectors, or, in other words, they can be overwhelmingly *long* and their structure can become highly complicated. This inherent complexity can probably be best explained by certain classes of two-colored graphs. It needs to be added that this approach goes back to Beck ([4]) and was rediscovered by Bilyk and Lacey in [11].

We will proceed as follows. First of all, we provide all necessary definitions and preliminaries to be able to work within our graph theoretic setting and state the connection to our actual problem. Secondly, we prove results in the spirit of the simple Beck gain, Lemma 2.32 before we finally verify (2.27).

Graph nomenclature and preliminaries

The structure of coincidences within collections of hyperbolic graphs can be very well explained by two-colored graphs. These are triples $G = (V, E_2, E_3)$, where $V \subseteq [\![q]\!]$ denotes the set of vertices and the symmetric subsets of $V \times V \setminus \{(k,k) : \ k \in V\}$, E_2 and E_3, are the edge sets of colors 2 and 3, respectively. Additionally, we say that $Q \subseteq V$ is a clique of color j iff it is subject to

$$\forall v, w \in Q, v \neq w : \ (v, w) \in E_j$$

and Q is maximal with this property. Here, maximality is understood in the sense that if $\tilde{Q} \supseteq Q$ fulfills the above condition, then $\tilde{Q} = Q$. Notice that edges serve to indicate that two vectors have a coincidence and its color states the coordinate. Hence, vertices from one clique of, say, color 2 shall correspond to a collection of hyperbolic vectors which have a coincidence in the second coordinate. Furthermore, we call a two-colored graph *connected* iff for all vertices $v, w \in V$ there exist vertices $v_1 = v, v_2, \ldots, v_{m-1}, v_m = w$ in V such that (v_j, v_{j+1}) lies in one of the two edge sets for each $1 \leq j < m$.

As a matter of fact, we only need to consider special types of graphs as set out in the definition below.

Definition 2.38. A two-colored graph G is called *admissible* iff the following four conditions are fulfilled:

(i) Each E_j decomposes into a union of cliques,

(ii) If Q_2 and Q_3 are cliques of color two and three, respectively, then $|Q_2 \cap Q_3| \in \{0, 1\}$.

(iii) Every vertex is contained in at least one clique.

(iv) Cliques of the same color are disjoint.

Moreover, we subdivide the class of admissible connected (a.c.) graphs on a given vertex set V further into $\mathcal{T}(V)$ and $\mathcal{C}(V)$. Here, $\mathcal{T}(V)$ comprises all a.c. graphs G defined on V such that either

(i) G is a tree

(ii) if G contains a cycle then this cycle is composed of edges of one color only,

and $\mathcal{C}(V)$ contains the rest. I.e., $\mathcal{C}(V)$ contains cycles composed of edges of both colors. We shall refer to such cycles as *bicolored*.

Items (i)–(iii) of the first enumeration in the above definition give an admissible graph in the sense of Bilyk and Lacey [11, p. 106]. The second item hereby reflects our hyperbolic assumption. Indeed, if two vectors from $\mathbb{H}_n^3$ have a coincidence in the second and the third coordinate, they are automatically equal. The third item guarantees that we only consider those components which are not strongly distinct.

Here, we extend the definition by Bilyk and Lacey by item (iv), which serves as some form of uniqueness of a representation. Consider a sufficiently large set of hyperbolic vectors which all share the same second coordinate, for instance. A corresponding admissible graph may now be drawn as a cycle, a tree, a graph comprising trees and cycles as subgraphs, two separate trees, etc. With this additional restriction we confine ourselves to one possible representation from which we only know that it is connected. That being said, observe that if we regard individual cliques as vertices themselves, the elements of $\mathcal{T}(V)$ admit of a tree representation. This is why we refer to them as *generalized trees* in all that follows.

A bound for the number of admissible graphs on a given vertex set is given in the lemma below.

Lemma 2.39 (Cf. [69, Lemma 5.2]). *Let $V \subseteq [\![q]\!]$. The number of admissible graphs on V is bounded by $c|V|^{2|V|}$, $c > 0$. For generalized tree graphs this number reduces to $2^{|V|}|V|^{|V|-2}$.*

Proof. The first bound is derived in [8, p. 144] and the estimate for generalized trees is better known as *Cayley's formula* without the additional factor $2^{|V|}$ which arises from choosing one of two colors for each edge. Since elements of $\mathcal{T}(V)$ can deviate from actual trees in a prescribed manner only (see item (iv) of Definition 2.38) this estimate continues to hold for generalized trees. Cayley's formula was originally shown by Borchardt in 1860. Four more recent proofs can be found in the book [1], for instance. □

Let us now draw the connection to our problem, i.e., the study of coincidences. To this end we define

$$\mathbb{X}(G) := \left\{ (\vec{r}_1, \vec{r}_2, \dots, \vec{r}_{|V|}) \in \prod_{v \in V} \mathbb{A}_v : \ (v_1, v_2) \in E_j \Rightarrow r_{v_1}^{(j)} = r_{v_2}^{(j)} \right\}, \quad (2.48)$$

where V is the vertex set on which G is defined and where $r_v^{(j)}$ denotes the j-th coordinate of the vector $\vec{r}_v$. Notice that the connection between edge sets and coincidences is not one-to-one, as the absence of an edge does not automatically imply that there is no coincidence. Hence, we require an auxiliary entity, the so-called *index* of an admissible grah G, i.e. $\mathrm{ind}(G)$.

According to [13, p. 2495] it indicates the number of coincidences needed to define $\mathbb{X}(G)$. The lemma below can now be derived using the inclusion-exclusion principle (cf. [13, (8.2)]).

Lemma 2.40. *The function $\Psi^{\neg}$ comprising the not strongly distinct part of Ψ can be rewritten as*

$$\Psi^{\neg} = \sum_{G \text{ admissible}} (-1)^{\mathrm{ind}(G)+1} \tilde{\rho}^{|V(G)|} \operatorname{SP}(\mathbb{X}(G)) \prod_{v \notin V(G)} (1 + \tilde{\rho} F_v) .$$

The Beck gain for graphs

We aim at finding good estimates for the L^p-norm of $\operatorname{SP}(\mathbb{X}(G))$ for admissible graphs G in the spirit of Lemma 2.32. In doing so we take another reduction step from admissible to connected admissible graphs. To this end, we cite another result by Bilyk and Lacey, [11, Proposition 10.7].

Lemma 2.41. *Let $G_1, G_2, \ldots, G_k$ be a.c. graphs defined on the mutually disjoint vertex sets $V_1, V_2, \ldots, V_k$. Let us regard these graphs as disjoint subgraphs of a supergraph G with vertex set $V(G) = \bigcup_{j=1}^{k} V_j$. We then have*

$$\rho^{|V(G)|} \operatorname{SP}(\mathbb{X}(G)) = \prod_{j=1}^{k} \rho^{|V_j|} \operatorname{SP}(\mathbb{X}(G_j)) .$$

In [11, p. 109] Bilyk and Lacey describe an algorithm for estimating $\| \operatorname{SP}(\mathbb{X}(G))\|_p$ for a.c. graphs G. We rely on its basic structure to derive (2.27). Generally speaking, we dispose of two tools for estimating the norm of $\operatorname{SP}(\mathbb{X}(G))$, i.e., the Littlewood–Paley inequality and the triangle inequality. Carrying out any of these steps naturally involves fixing the value of a certain entry of a specific vector. To provide the reader with a clearer picture of the arguments used in the proof of 2.42 we present an introductory example.

$\vec{r}$		$\vec{s}$		$\vec{t}$
r_1		s_1		t_1
r_2	$=$	s_2	$\neq$	t_2
r_3	$\neq$	s_3	$=$	t_3

$\vec{r}$		$\vec{s}$		$\vec{t}$
r_1		s_1		μ
r_2	$=$	s_2	$\neq$	ν
r_3	$\neq$	$n-\mu-\nu$	$=$	$n-\mu-\nu$

Figure 2.11: Hyperbolic vectors associated to the graphs G_0 (left) and $\tilde{G}_0$ (right).

Let us consider the graph G_0 defined on three vertices associated to the first picture in Figure 2.11. Recall that the first coordinates of these vectors are necessarily distinct, since each of them belongs to a different set of the partition $I_1, I_2, \dots, I_q$ of $[\![n]\!]$ which we introduced in (2.21). Hence, the Littlewood–Paley inequality is applicable in the first coordinate. We assume w.l.o.g. $t_1 \in I_1$ and obtain

$$\| \operatorname{SP}(\mathbb{X}(G_0))\|_p = \Bigg\| \sum_{(\vec{r},\vec{s},\vec{t})\in\mathbb{X}(G_0)} f_{\vec{r}} f_{\vec{s}} f_{\vec{t}} \Bigg\|_p \lesssim p^{\frac{1}{2}} \Bigg\| \Bigg[\sum_{\mu\in I_1} \Bigg| \sum_{\substack{(\vec{r},\vec{s},\vec{t})\in\mathbb{X}(G_0)\\ \vec{t}=(\mu,t_2,t_3)}} f_{\vec{r}} f_{\vec{s}} f_{\vec{t}} \Bigg|^2 \Bigg]^{\frac{1}{2}} \Bigg\|_p .$$

Subsequently, we fix t_2 with the help of the triangle inequality

$$\| \operatorname{SP}(\mathbb{X}(G_0))\|_p \lesssim p^{\frac{1}{2}} \sum_{\nu=1}^{n} \Bigg\| \Bigg[\sum_{\mu\in I_1} \Bigg| \sum_{\substack{(\vec{r},\vec{s},\vec{t})\in\mathbb{X}(G_0)\\ \vec{t}=(\mu,\nu,t_3)}} f_{\vec{r}} f_{\vec{s}} f_{\vec{t}} \Bigg|^2 \Bigg]^{\frac{1}{2}} \Bigg\|_p .$$

Notice that $\vec{t}$ is already fully specified, since its coordinates add up to n. Consequently, we can pull it out of the sum and use $f_{\vec{t}}^2 =\equiv 1$. Taking the supremum w.r.t. μ and ν then finally yields

$$\| \operatorname{SP}(\mathbb{X}(G_0))\|_p \lesssim p^{\frac{1}{2}} q^{-\frac{1}{2}} n^{\frac{3}{2}} \sup_{\mu,\nu} \| \operatorname{SP}(\mathbb{X}(\tilde{G}_0))\|_p,$$

where we used $|I_1| = n/q$. The vectors from $\mathbb{X}(\tilde{G}_0)$ are depicted in the right picture of Figure 2.11. Observe that we only need to carry out the first of the above steps, i.e. the Littlewood–Paley inequality, in order to completely determine $\vec{s}$. We continue in this direction until we have considered every vertex as then the expression in modulus equals to 1.

Lemma 2.42 (Beck gain for long coincidences, cf. [11, Theorem 10.1], [69, Lemma 5.3])**.** *Let G be an a.c. graph with vertex set V, $|V| \geq 2$, comprising exactly t disjoint bicolored cycles. We have*

$$\tilde{\rho}^{|V|} \| \operatorname{SP}(\mathbb{X}(G))\|_p \lesssim p^{\frac{3}{2}} q^{2b-\frac{1}{4}} n^{-\frac{1}{2}}, \qquad \text{if } |V| = 2,$$

for all $p \leq q^{3/2} n$, and

$$\tilde{\rho}^{|V|} \| \operatorname{SP}(\mathbb{X}(G))\|_p \lesssim p^{\frac{|V|}{2}-\frac{t}{2}} q^{\frac{|V|(2b-1)}{2}+\frac{t}{2}} n^{-\frac{|V|}{2}+1-\frac{t}{2}}, \qquad \text{if } |V| \geq 3.$$

Moreover, in the special case where $p = lq^{2b}$, $1 \leq l \leq q$, these two estimates can be merged into

$$\begin{aligned}\tilde{\rho}^{|V|} \| \operatorname{SP}(\mathbb{X}(G))\|_{lq^{2b}} &\lesssim \min\left\{ l^{\frac{3}{2}} q^{5b-\frac{1}{4}} n^{-\frac{1}{2}}, l^{\frac{|V|}{2}} q^{\frac{|V|(4b-1)}{2}} n^{1-\frac{|V|}{2}} \right\} l^{-\frac{t}{2}} q^{\frac{t(1-2b)}{2}} n^{-\frac{t}{2}} \\ &=: M_{|V|,l}\, l^{-\frac{t}{2}} q^{\frac{t(1-2b)}{2}} n^{-\frac{t}{2}}\end{aligned}$$

for all $\varepsilon < \frac{2}{5+4b}$.

Proof. Noticing that bicolored cycles are not possible for $|V| = 2$ the first bound is merely a reformulation of the Beck gain in the simplest instance, i.e. the second estimate in Lemma 2.32. The second estimate follows the strategy outlined in the paragraphs preceding this lemma.

To this end let $V = \{v_1, v_2, \dots, v_k\}$, $k \geq 3$, and let us first assume $G \in \mathcal{T}(V)$, i.e. $t = 0$. We denote by $\mathbb{X}(G; \mu)$ those $(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_k) \in \mathbb{X}(G)$ for which the first component of $\vec{r}_k$, i.e. $r_k^{(1)}$, is equal to μ. One application of the Littlewood–Paley inequality in the first coordinate now yields (we assume w.l.o.g. that $r_k^{(1)} \in I_k$)

$$\left\| \mathrm{SP}\left(\mathbb{X}(G)\right) \right\|_p \lesssim p^{\frac{1}{2}} \left\| \left[\sum_{\mu \in I_k} \left| \mathrm{SP}\left(\mathbb{X}(G;\mu)\right)\right|^2 \right]^{\frac{1}{2}} \right\|_p . \tag{2.49}$$

In order to specify $\vec{r}_k$ completely we still need one more of its components to be fixed. With the help of the triangle inequality we set $r_k^{(2)} = \nu$, for instance. In doing so, we obtain the set $\mathbb{X}(G; \mu; \nu)$ consisting of tuples $(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_{k-1})$ such that $\{\vec{r}_1, \dots, \vec{r}_{k-1}, (\mu, \nu, n - \mu - \nu)\} \in \mathbb{X}(G; \mu)$. This discussion results in

$$\begin{aligned} \left\| \mathrm{SP}\left(\mathbb{X}(G)\right) \right\|_p &\lesssim p^{\frac{1}{2}} \sum_{\nu=1}^{n} \left\| \left[\sum_{\mu \in I_k} \left| \mathrm{SP}\left(\mathbb{X}(G;\mu;\nu)\right)\right|^2 \right]^{\frac{1}{2}} \right\|_p \\ &\lesssim p^{\frac{1}{2}} n \sup_{\nu} \left\| \left[\sum_{\mu \in I_k} \left| \mathrm{SP}\left(\mathbb{X}(G;\mu;\nu)\right)\right|^2 \right]^{\frac{1}{2}} \right\|_p . \end{aligned} \tag{2.50}$$

Taking the supremum over all μ too finally yields

$$\left\| \mathrm{SP}\left(\mathbb{X}(G)\right) \right\|_p \lesssim p^{\frac{1}{2}} q^{-\frac{1}{2}} n^{\frac{3}{2}} \sup_{\mu,\nu} \left\| \mathrm{SP}\left(\mathbb{X}(G;\mu;\nu)\right) \right\|_p . \tag{2.51}$$

Since G is connected, we automatically fix one coordinate of an adjacent vertex whenever we fully determine one vector. By the hyperbolic assumption, this adjacent vertex in turn is then fully determined if we fix its first coordinate with the help of the Littlewood–Paley inequality and taking the supremum again, as we did above. Observe that in doing so we avoid the supplementary application of the triangle inequality and therefore each such additional step can be carried out at a cost of $p^{1/2} q^{-1/2} n^{1/2}$. Finally, this procedure terminates after having considered all vertices, as then

$$\left| \mathrm{SP}\left(\mathbb{X}(G; \vec{\mu}; \nu)\right) \right| \equiv 1,$$

where $\vec{\mu}$ contains all the fixed first coordinates and where $\mathbb{X}(G;\vec{\mu};\nu)$ is defined analogously as above. In the end we obtain

$$\begin{aligned}\|\mathrm{SP}\left(\mathbb{X}(G)\right)\|_p &\lesssim p^{\frac{1}{2}}q^{-\frac{1}{2}}n^{\frac{3}{2}}\left(p^{\frac{1}{2}}q^{-\frac{1}{2}}n^{\frac{1}{2}}\right)^{|V|-1}\sup_{\vec{\mu},\nu}\|\mathbb{X}(G;\vec{\mu};\nu)\|_p\\ &= p^{\frac{|V|}{2}}q^{-\frac{|V|}{2}}n^{\frac{|V|}{2}+1}.\end{aligned}$$

Let now $G \in \mathcal{C}(V)$ with t disjoint bicolored cycles. Notice that one can find a vertex where edges of color two and three meet within each such cycle. Therefore, the underlying vector is fully determined by its neighboring vectors. Revisiting the proof in the generalized tree case thus shows that this vertex does not need to be considered in the algorithm explained above. Consequently, we save one application of the Littlewood–Paley inequality and, hence, save a factor of $p^{1/2}q^{-1/2}n^{1/2}$ for each of the t disjoint bicolored cycles compared to a generalized tree with the same number of vertices. This proves the second assertion.

Finally, we specialize $p = lq^{2b}$, $1 \leq l \leq q$. Inserting this value for p in the first two bounds of the claim gives the entries from within the minimum as well as the correction factor in case of the presence of bicolored cycles. We still need to proof that taking the minimum of the two terms is justified as well as to check the bound for ε. If $|V| = 2$ the first entry is smaller. Indeed, the first expression outweighs the second one iff

$$l^{\frac{3}{2}}q^{5b-\frac{1}{4}}n^{-\frac{1}{2}} \lesssim lq^{4b-1} \quad \Leftrightarrow \quad l^{\frac{1}{2}}q^{b+\frac{3}{4}} \lesssim n^{\frac{1}{2}}.$$

Evidently, this is the case if $\varepsilon < 2/(4b+5)$, as $l \leq q$. On the other hand, for $|V| \geq 3$ we have

$$l^{\frac{|V|}{2}}q^{\frac{|V|(4b-1)}{2}}n^{1-\frac{|V|}{2}} \leq l^{\frac{3}{2}}q^{5b-\frac{1}{4}}n^{-\frac{1}{2}} \quad \Leftrightarrow \quad l^{\frac{|V|-3}{2}}q^{\frac{|V|(4b-1)}{2}-5b+\frac{1}{4}} \leq n^{\frac{|V|-3}{2}},$$

which in turn holds whenever

$$\left(2b|V| - 5b - \frac{5}{4}\right)\varepsilon \leq \frac{|V|-3}{2}.$$

We are done if the right-hand side is negative. If this is not the case, we observe that any $\varepsilon < 1/(4b)$ satisfies

$$\left(2b|V| - 5b - \frac{5}{4}\right)\varepsilon \leq \frac{2|V|-5}{4} - \frac{5}{16b} < \frac{2|V|-5}{4} - \frac{1}{4} = \frac{|V|-3}{2}.$$

Finally, we conclude the proof by noticing that

$$\frac{1}{4b} = \frac{2}{8b} > \frac{2}{5+4b}.$$

□

Now, our strategy becomes more visible. While a.c. graphs with bicolored cycles are hard to handle combinatorically speaking (see Lemma 2.39), they yield much better estimates in terms of Lemma 2.42 compared to graphs from $\mathcal{T}$. As it turns out in the subsequent part of this section, generalized trees account for the lion share in our estimates.

The proof of (2.27)

In order to verify the asserted inequality (2.27) we once again use the same starting point as Bilyk and Lacey did in [11]. As soon as the more specialized version of the Beck gain (Lemma 2.42) can be incorporated we follow the ideas from the author's paper [69].

For the actual proof we require one more technical lemma given below. Within its proof we use another (interpolating) version of Hölder's inequality, which is sometimes also referred to as *Littlewood's inequality.* Let $\theta \in (0,1)$ and p_1 and p_2 such that

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}.$$

Then,

$$\|f\|_p \leq \|f\|_{p_1}^{\theta} \|f\|_{p_2}^{1-\theta}. \tag{2.52}$$

Lemma 2.43 (Cf. [11, Lemma 9.1 and p. 96])**.** *Let $0 < b < 1/4$. For F_v, $1 \leq v \leq q$, as introduced in Section 2.3.3 and all $\varepsilon < \min\{1/3, 1/(1+12b)\}$ we have*

a)

$$\sup_{V \subseteq \{1,2,\dots,q\}} \mathbb{E} \prod_{v \in V} (1 + \tilde{\rho} F_v)^2 \lesssim \exp(a^2 q^{2b}),$$

and

b)

$$\sup_{V \subseteq \{1,2,\dots,q\}} \Big\| \prod_{v \in V} (1 + \tilde{\rho} F_v) \Big\|_{q^{2b}/(q^{2b}-1)} \lesssim 1.$$

Proof. First off we notice that the proof of Lemma 2.37 can be carried out for an arbitrary subset of $\{1,2,\dots,q\}$. This leads to the uniform estimate a). The upper bound for ε in this lemma is $\min\{1/3, 1/(1+12b)\}$.

The estimate b) follows from Littlewood's inequality. Indeed, choose $\theta = (q^{2b}-2)/q^{2b}$ as well as $p_1 = 1$ and $p_2 = 2$. Then,

$$\sup_V \Big\| \prod_{v \in V} (1 + \tilde{\rho} F_v) \Big\|_{q^{2b}/(q^{2b}-1)} \leq \sup_V \Big\| \prod_{v \in V} (1 + \tilde{\rho} F_v) \Big\|_1^{\frac{q^{2b}-2}{q^{2b}}} \Big\| \prod_{v \in V} (1 + \tilde{\rho} F_v) \Big\|_2^{\frac{2}{q^{2b}}}.$$

The first factor can be estimated by the L^1-norm of the Riesz product Ψ, which is given in (2.26) and proved in Section 2.3.5, i.e., $\|\Psi\|_1 \lesssim 1$ for b and ε as stated in the claim. The second factor is controlled by the estimate from a). Thus, we obtain further

$$\sup_V \left\| \prod_{v\in V} (1+\tilde{\rho}F_v) \right\|_{q^{2b}/(q^{2b}-1)} \leq \exp\left(a^2 q^{2b} \frac{1}{q^{2b}}\right) \simeq 1.$$

□

In order to prove $\|\Psi^{\neg}\|_1 \lesssim 1$ we may now proceed as follows (cp. [13, p. 2481]). We apply Lemma 2.40, Hölder's inequality, and the second item from Lemma 2.43 in the third step to obtain

$$\begin{aligned}
\|\Psi^{\neg}\|_1 &\leq \sum_{G \text{ admissible}} \left\| \tilde{\rho}^{|V(G)|}\,\mathrm{SP}(\mathbb{X}(G)) \prod_{v\in\{1,\dots,q\}\setminus V(G)} (1+\tilde{\rho}F_v) \right\|_1 \\
&\leq \sum_{G \text{ admissible}} \left\| \tilde{\rho}^{|V(G)|}\,\mathrm{SP}(\mathbb{X}(G)) \right\|_{q^{2b}} \left\| \prod_{v\in\{1,\dots,q\}\setminus V(G)} (1+\tilde{\rho}F_v) \right\|_{q^{2b}/(q^{2b}-1)} \\
&\leq \sum_{G \text{ admissible}} \left\| \tilde{\rho}^{|V(G)|}\,\mathrm{SP}(\mathbb{X}(G)) \right\|_{q^{2b}}
\end{aligned}$$

for all $\varepsilon \leq \min\{1/3, 1/(1+12b)\}$. Notice that the summands now only depend on the cardinality of $V(G)$, not on the particular vertices. Hence, we may rewrite

$$\sum_{G \text{ admissible}} \left\| \tilde{\rho}^{|V(G)|}\,\mathrm{SP}(\mathbb{X}(G)) \right\|_{q^{2b}} = \sum_{v=2}^{q} \sum_{\substack{G\text{admissible} \\ |V(G)|=v}} \left\| \tilde{\rho}^{v}\,\mathrm{SP}(\mathbb{X}(G)) \right\|_{q^{2b}}. \quad (2.53)$$

For $|V|=2$ there exist $2\binom{q}{2} \simeq q^2$ different graphs, each of which satisfies the corresponding bound from Lemma 2.42, i.e., the first entry in the minimum with $l=1$and $t=0$. Hence, the first summand in (2.53) is bounded by

$$q^2 q^{5b-\frac{1}{4}} n^{-\frac{1}{2}} \simeq n^{\frac{20b+7}{4}\varepsilon-\frac{1}{2}} \lesssim 1 \quad \text{if} \quad \varepsilon < \frac{2}{20b+7}.$$

Observe that the bound for ε dictated by the above estimate is dominated by the one required to apply Lemma 2.42. If $|V|=3$, still, every a.c. graph belongs to $\mathcal{T}(V)$, due to the hyperbolic assumption. Similarly as above, we see that, up to constant factors, there are q^3 different graphs in this case and together with the Beck gain for graphs ($l=1$, $t=0$) we obtain that the summand for $v=3$ satisfies

$$q^3 q^{6b-\frac{3}{2}} n^{1-\frac{3}{2}} \simeq n^{\frac{12b+3}{2}\varepsilon-\frac{1}{2}} \lesssim 1 \quad \text{if} \quad \varepsilon < \frac{1}{12b+3}.$$

Notice that the bounds for ε coincide at the critical value $b = 1/4$ where they evaluate to $1/6$.

In what follows we abbreviate $\binom{W}{k} := \{U \subseteq W : \ |U| = k\}$ for positive integers k and finite sets W. We have the following lemma below.

Lemma 2.44. *For $V \subseteq [\![q]\!]$ and $1 \le l \le q$ let*

$$\mathcal{V}(V,l) = \{\mathbf{V} = (V_1, V_2, \dots, V_l) : \ \mathbf{V} \text{ is a partition of } V\}.$$

The cardinality of this set is bounded by

$$\#\mathcal{V}(V,l) \le \frac{1}{2}\binom{|V|}{l} l^{|V|-l}.$$

Proof. The sought number is also known as the so-called *Stirling number of the second kind*. A proof for this inequality is given in [70]. □

To pursue our strategy, i.e. balancing the advantageous combinatorial aspects of generalized trees against the higher gain of graphs from $\mathcal{C}$ in Lemma 2.42, we need to split up the remaining sum in (2.53) into considerably more detailled components. To this end, we write each admissible graph G as a union of its connected components. Subsequently, we exploit Lemma 2.41 together with the generalized Hölder's inequality, giving

$$\begin{aligned} \|\Psi^{\neg}\|_1 &\lesssim \sum_{v=4}^{q} \sum_{V \in \binom{[\![q]\!]}{v}} \sum_{l=1}^{v/2} \sum_{(V_1,\dots,V_l) \in \mathcal{V}(V,l)} \sum_{\substack{G = G_1 \cup \dots \cup G_l \\ G_j \text{ is a.c. on } V_j}} \prod_{j=1}^{l} \tilde{\rho}^{|V_j|} \left\| \mathrm{SP}(\mathbb{X}(G_j)) \right\|_{l^{q^{2b}}} \\ &=: \sum_{v=4}^{q} \sum_{V \in \binom{[\![q]\!]}{v}} \sum_{l=1}^{v/2} \sum_{(V_1,\dots,V_l) \in \mathcal{V}(V,l)} \left(\Sigma_{\text{tree}} + \Sigma_{\text{cycle}}\right), \end{aligned} \tag{2.54}$$

where

$$\Sigma_{\text{tree}} = \sum_{\substack{G = G_1 \cup \dots \cup G_l \\ G_j \in \mathcal{T}(V_j)}} \prod_{j=1}^{l} \tilde{\rho}^{|V_j|} \left\| \mathrm{SP}(\mathbb{X}(G_j)) \right\|_{l^{q^{2b}}} \quad \text{and}$$

$$\Sigma_{\text{cycle}} = \sum_{\substack{G = G_1 \cup \dots \cup G_l \\ G_j \text{ a.c. on } V_j \text{ and } \exists j_0 : T(G_{j_0}) \ge 1}} \prod_{j=1}^{l} \tilde{\rho}^{|V_j|} \left\| \mathrm{SP}(\mathbb{X}(G_j)) \right\|_{l^{q^{2b}}}.$$

with $T(G_j) = \max\{\tau : G_j \text{ contains } \tau \text{ disjoint bicolored cycles}\}$.

Before we continue with estimating Σ_{tree} we shall give one more technical lemma.

Lemma 2.45 (Cf. [69, Lemma 5.4]). *Let l, k, and v be integers with $1 \leq k \leq l \leq v/2$. Furthermore, consider $v_1, v_2, \dots, v_l \in \mathbb{N}$ with $v_j \geq 2$, $1 \leq j \leq l$, and $v_1 + v_2 + \dots + v_l = v$. Then*

$$\left(\prod_{j=1}^{k} v_j^{v_j-2}\right) \cdot \left(\prod_{j=k+1}^{l} v_j^{2v_j}\right) \lesssim \left(\frac{v}{k}\right)^{v-2k},$$

and if $k = 0$, i.e., the first product vanishes, we obtain

$$\prod_{j=1}^{l} v_j^{2v_j} \leq \left(\frac{v}{l}\right)^{2v}.$$

Proof. We confine ourselves to the case where $k \geq 1$, since the proof for $k = 0$ follows the same spirit and the result is even easier to obtain. Let us consider the logarithm of the left-hand side and maximize it with respect to the constraint $v_1 + v_2 + \dots + v_l = v$. This approach leads to the Lagrangian

$$\mathcal{L}(v_1, v_2, \dots, v_l; \lambda) = \sum_{j=1}^{k} (v_j - 2) \log v_j + 2 \sum_{j=k+1}^{l} v_j \log v_j - \lambda(v_1 + v_2 + \dots + v_l - v).$$

Setting all first partial derivatives with respect to some v_j equal to zero and solving the corresponding system of equations yields

$$v_j = \frac{2}{w}, \quad 1 \leq j \leq k \quad \text{and} \quad v_j = \mathrm{e}^{\frac{\lambda}{2}-1}, \quad k < j \leq l,$$

where $w = W(2\mathrm{e}^{1-\lambda})$ with W denoting the *Lambert W function*. That is, $W(z) = w$ iff $z = w\mathrm{e}^w$. In order to determine the critical value for λ we notice that

$$2\mathrm{e}^{1-\lambda} = w\mathrm{e}^w \quad \Longleftrightarrow \quad \mathrm{e}^{\frac{\lambda}{2}-1} = \frac{w}{2}\mathrm{e}^{w+\frac{3\lambda}{2}-2},$$

and, hence,

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{2k}{w} + (l-k)\mathrm{e}^{\frac{\lambda}{2}-1} - v = \frac{2k}{w} + \frac{w(l-k)}{2}\mathrm{e}^{w+\frac{3\lambda}{2}-2} - v.$$

This is equal to zero iff

$$\lambda = \frac{2}{3}\left(2 + \log\left(\frac{2(wv-2k)}{\mathrm{e}^w w^2 (l-k)}\right)\right) = \frac{2}{3}\left(2 + \log\left(\frac{wv-2k}{\mathrm{e}^{1-\lambda} w (l-k)}\right)\right) \tag{2.55}$$

$$= \frac{2}{3}\left(2 + \log\left(\frac{\mathrm{e}^{1-\lambda} v - k\mathrm{e}^w}{\mathrm{e}^{2-2\lambda}(l-k)}\right)\right). \tag{2.56}$$

Let us first assume that $1 \le j \le k$. In this case we observe that $0 < w \le 1$, since $v_j \ge 2$, and, consequently,

$$v_j = \frac{2}{w} \le \frac{2}{w}\mathrm{e}^{1-w} = \frac{2\mathrm{e}}{w\mathrm{e}^w} = \mathrm{e}^{\lambda}. \tag{2.57}$$

In the other case we aim for the upper bound

$$v_j = 2\mathrm{e}^{\frac{\lambda}{2}-1} \lesssim \left(\frac{v}{k}\right)^{\frac{1}{2}}, \qquad k < j \le l. \tag{2.58}$$

To this end we estimate as follows on the basis of (2.55)

$$\frac{3}{2}\lambda - 2 = \lambda - 1 + \log\left(\frac{v}{l-k} - \frac{2k}{w(l-k)}\right) \le \lambda - 1 + \log\frac{v}{l-k}.$$

If, additionally, $l - k > \sqrt{vk}$ we therefore obtain

$$\mathrm{e}^{\frac{\lambda}{2}-1} \le \frac{v}{l-k} < \left(\frac{v}{k}\right)^{\frac{1}{2}}.$$

Conversely, if $l - k \le \sqrt{vk}$ we solve (2.56) for λ (pretending that w is independent of λ), giving

$$\lambda = 2\log\left(\frac{l-k+\sqrt{(l-k)^2+4\mathrm{e}^{w+1}vk}}{2k\mathrm{e}^w}\right).$$

We thus obtain without difficulty

$$\mathrm{e}^{\frac{\lambda}{2}} \le \frac{l-k+\sqrt{(l-k)^2+4\mathrm{e}^{w+1}vk}}{2k\mathrm{e}^w} \le \frac{(l-k)+\mathrm{e}^{\frac{w+1}{2}}\sqrt{vk}}{k\mathrm{e}^w} \lesssim \left(\frac{v}{k}\right)^{\frac{1}{2}},$$

where we used the subadditivity of the square root in the first and our assumption $l - k \le \sqrt{vk}$ as well as $\mathrm{e}^w \simeq 1$ in the second step. This completes the proof of (2.58).

Altogether we obtain by (2.57) and (2.58)

$$\begin{aligned}\left(\prod_{j=1}^{k} v_j^{v_j-2}\right)\cdot\left(\prod_{j=k+1}^{l} v_j^{2v_j}\right) &\le \mathrm{e}^{\lambda(v_1+\cdots+v_k-2k)}\mathrm{e}^{2(\frac{\lambda}{2}-1)(v_{k+1}+\cdots+v_l)}\\ &\le \mathrm{e}^{\lambda(v_1+\cdots+v_l)-2\lambda k} = \mathrm{e}^{\lambda(v-2k)} \lesssim \left(\frac{v}{k}\right)^{v-2k}.\end{aligned}$$

□

Within the subsequent paragraphs we show

$$\sum_{v=4}^{q} \sum_{V \in \binom{[\![q]\!]}{v}} \sum_{l=1}^{v/2} \sum_{(V_1,\dots V_l)\in\mathcal{V}(V,l)} \Sigma_{\text{tree}} \lesssim 1 \tag{2.59}$$

for all $\varepsilon < \varepsilon^\tau(b)$. For fixed $4 \le v \le q$, $V \in \binom{[\![q]\!]}{v}$, $l \le v/2$, and $(V_1, \dots, V_l) \in \mathcal{V}(V,l)$ we obtain

$$\begin{aligned}\Sigma_{\text{tree}} &= \prod_{j=1}^{l} \sum_{G_j \in \mathcal{T}(V_j)} \|\mathrm{SP}(\mathbb{X}(G_j))\|_{l_{q^{2b}}} \\ &\lesssim \prod_{j=1}^{l} M_{|V_j|,l} \sum_{G_j\in\mathcal{T}(V_j)} 1 \lesssim \prod_{j=1}^{l} M_{|V_j|,l} |V_j|^{|V_j|-2}\end{aligned}$$

for all $\varepsilon < 2/(5+4b)$. Here, we used Lemma 2.42 with $t = 0$ in the second and Cayley's formula, i.e. the second estimate from Lemma 2.39, in the third step. One application of Lemma 2.45 with $k = l$ thus leads to

$$\Sigma_{\text{tree}} \lesssim l^{-v+2l} v^{v-2l} \prod_{j=1}^{l} M_{|V_j|,l}. \tag{2.60}$$

Let us choose $\alpha_\tau \in (0, 1/2)$ arbitrarily for now and consider the sum over l. The exact value for α_τ shall be determined at the end of the proof of (2.59). Observe that, if l, i.e. the number of sets V_j in a specific partition of V, is small, then the individual sets of the partition are more likely to be large and vice versa. Hence, we choose the second entry of the minimum $M_{|V_j|,l}$ for the first $\alpha_\tau v$ summands and the first entry for the others. In doing so we obtain

$$\begin{aligned}\sum_{l=1}^{v/2} \sum_{(V_1,\dots,V_l)\in\mathcal{V}(V,l)} \Sigma_{\text{tree}} \lesssim &\sum_{l=1}^{\alpha_\tau v} \sum_{(V_1,\dots,V_l)\in\mathcal{V}(V,l)} l^{-v+2l} v^{v-2l} \prod_{j=1}^{l} l^{\frac{|V_j|}{2}} q^{\frac{|V_j|(4b-1)}{2}} n^{-\frac{|V_j|+1}{2}} \\ &+ \sum_{l=\alpha_\tau v+1}^{v/2} \sum_{(V_1,\dots,V_l)\in\mathcal{V}(V,l)} l^{-v+2l} v^{v-2l} \prod_{j=1}^{l} l^{\frac{3}{2}} q^{5b-\frac{1}{4}} n^{-\frac{1}{2}} \\ \lesssim &\sum_{l=1}^{\alpha_\tau v} \binom{v}{l} l^{\frac{v}{2}+l} v^{v-2l} q^{\frac{4b-1}{2}v} n^{-\frac{v}{2}+l} \\ &+ \sum_{l=\alpha_\tau v+1}^{v/2} \binom{v}{l} l^{\frac{5}{2}l} v^{v-2l} q^{(5b-\frac{1}{4})l} n^{-\frac{l}{2}} \\ =: &\Sigma_1^\tau + \Sigma_2^\tau,\end{aligned} \tag{2.61}$$

where we used Lemma 2.44 in the last step. Recall that $l! \simeq l^{l+1/2}$ by Stirling's formula and, therefore,

$$\begin{aligned}\Sigma_1^\tau &\leq \sum_{l=1}^{\alpha_\tau v} \frac{v^l}{l!} l^{\frac{v}{2}+l} v^{v-2l} q^{(2b-\frac{1}{2})v} n^{-\frac{v}{2}+l} \simeq \sum_{l=1}^{\alpha_\tau v} l^{\frac{v}{2}-\frac{1}{2}} v^{v-l} q^{(2b-\frac{1}{2})v} n^{-\frac{v}{2}+l} \\ &\leq v^{\frac{3}{2}v-\frac{1}{2}} q^{(2b-\frac{1}{2})v} n^{-\frac{v}{2}} \sum_{l=1}^{\alpha_\tau v} \left(v^{-1}n\right)^l \lesssim v^{(\frac{3}{2}-\alpha_\tau)v-\frac{1}{2}} q^{(2b-\frac{1}{2})v} n^{-(\frac{1}{2}-\alpha_\tau)v}. \qquad (2.62)\end{aligned}$$

Furthermore, the second sum Σ_2^τ can be rewritten such that it starts at $l = 0$

$$\begin{aligned}\Sigma_2^\tau = \sum_{l=0}^{\frac{v}{2}-\alpha_\tau v-1} \binom{v}{l+\alpha_\tau v+1} (l+\alpha_\tau v+1)^{\frac{5}{2}(l+\alpha_\tau v+1)} \\ \times v^{v-2(l+\alpha_\tau v+1)} q^{(5b-\frac{1}{4})(l+\alpha_\tau v+1)} n^{-\frac{1}{2}(l+\alpha_\tau v+1)}.\end{aligned}$$

Observe that

$$\begin{aligned}\binom{v}{l+\alpha_\tau v+1} &= \binom{v-\alpha_\tau v-1}{l} \frac{l!}{(l+\alpha_\tau v+1)!} (v-\alpha_\tau v)(v-\alpha_\tau v+1)\cdots v \\ &\lesssim \binom{v-\alpha_\tau v-1}{l} l^{l+\frac{1}{2}} (l+\alpha_\tau v+1)^{-l-\alpha_\tau v-\frac{3}{2}} v^{\alpha_\tau v+1} \\ &\leq \binom{v-\alpha_\tau v-1}{l} (l+\alpha_\tau v+1)^{-\alpha_\tau v-1} v^{\alpha_\tau v+1}. \qquad (2.63)\end{aligned}$$

Consequently,

$$\begin{aligned}\Sigma_2^\tau \lesssim \sum_{l=0}^{\frac{v}{2}-\alpha_\tau v-1} \binom{v-\alpha_\tau v-1}{l} (l+\alpha_\tau v+1)^{\frac{3}{2}\alpha_\tau v+\frac{5}{2}l+\frac{3}{2}} \qquad (2.64) \\ \times v^{(1-\alpha_\tau)v-2l-1} q^{(5b-\frac{1}{4})(\alpha_\tau v+l+1)} n^{-\frac{1}{2}(\alpha_\tau v+l+1)} \\ \simeq v^{(1+\frac{\alpha_\tau}{2})v+\frac{1}{2}} q^{(5b-\frac{1}{4})(\alpha_\tau v+1)} n^{-\frac{1}{2}(\alpha_\tau v+1)} \sum_{l=0}^{\frac{v}{2}-\alpha_\tau v-1} \binom{v-\alpha_\tau v-1}{l} \left(v^{\frac{1}{2}} q^{5b-\frac{1}{4}} n^{-\frac{1}{2}}\right)^l \\ \leq v^{(1+\frac{\alpha_\tau}{2})v+\frac{1}{2}} q^{(5b-\frac{1}{4})(\alpha_\tau v+1)} n^{-\frac{1}{2}(\alpha_\tau v+1)} \left(1+v^{\frac{1}{2}} q^{5b-\frac{1}{4}} n^{-\frac{1}{2}}\right)^{v-\alpha_\tau v-1} \\ \lesssim v^{(1+\frac{\alpha_\tau}{2})v+\frac{1}{2}} q^{(5b-\frac{1}{4})(\alpha_\tau v+1)} n^{-\frac{1}{2}(\alpha_\tau v+1)} \mathrm{e}^{v^{3/2} q^{5b-1/4} n^{-1/2}} \\ \lesssim v^{(1+\frac{\alpha_\tau}{2})v+\frac{1}{2}} q^{(5b-\frac{1}{4})(\alpha_\tau v+1)} n^{-\frac{1}{2}(\alpha_\tau v+1)}, \qquad (2.65)\end{aligned}$$

since $\exp(v^{3/2} q^{5b-1/4} n^{-1/2}) \lesssim 1$ for all $\varepsilon < 2/(20b+5)$.

In the same spirit we can now prove (2.59):

$$
\begin{aligned}
\sum_{v=4}^{q} \sum_{V \in \binom{[\![q]\!]}{v}} (\Sigma_1^\tau + \Sigma_2^\tau) &\lesssim \sum_{v=4}^{q} \binom{q}{v} \Big(v^{(\frac{3}{2}-\alpha_\tau)v-\frac{1}{2}} q^{(2b-\frac{1}{2})v} n^{-(\frac{1}{2}-\alpha_\tau)v} \\
&\qquad + v^{(1+\frac{\alpha_\tau}{2})v+\frac{1}{2}} q^{(5b-\frac{1}{4})(\alpha_\tau v+1)} n^{-\frac{1}{2}(\alpha_\tau v+1)} \Big) \\
&\lesssim \sum_{v=0}^{q-4} \binom{q-4}{v} \left(\frac{q}{v+1}\right)^4 \Big((v+4)^{((\frac{3}{2}-\alpha_\tau)v+\frac{11}{2}-4\alpha_\tau} q^{(2b-\frac{1}{2})v+8b-2} n^{-(\frac{1}{2}-\alpha_\tau)v-2+4\alpha_\tau} \\
&\qquad + (v+4)^{(1+\frac{\alpha_\tau}{2})v+\frac{9}{2}+2\alpha_\tau} q^{(5b-\frac{1}{4})\alpha_\tau v+5b-\frac{1}{4}+(20b-1)\alpha_\tau} n^{-\frac{\alpha_\tau v}{2}-\frac{1}{2}-2\alpha_\tau} \Big) \\
&\lesssim \sum_{v=0}^{q-4} \binom{q-4}{v} \Big(q^{(1+2b-\alpha_\tau)v+\frac{7}{2}+8b-4\alpha_\tau} n^{-v(\frac{1}{2}-\alpha_\tau)-2+4\alpha_\tau} \\
&\qquad + q^{(1+(\frac{1}{4}+5b)\alpha_\tau)v+\frac{17}{4}+5b+(1+20b)\alpha_\tau} n^{-\frac{\alpha_\tau v}{2}-\frac{1}{2}-2\alpha_\tau} \Big) \\
&= q^{\frac{7}{2}+8b-4\alpha_\tau} n^{-2+4\alpha_\tau} \left(1 + q^{1+2b-\alpha_\tau} n^{-\frac{1}{2}+\alpha_\tau}\right)^{q-4} \\
&\qquad + q^{\frac{17}{4}+5b+(1+20b)\alpha_\tau} n^{-\frac{1}{2}-2\alpha_\tau} \left(1 + q^{1+(\frac{1}{4}+5b)\alpha_\tau} n^{-\frac{\alpha_\tau}{2}}\right)^{q-4} \\
&\lesssim q^{\frac{7}{2}+8b-4\alpha_\tau} n^{-2+4\alpha_\tau} \mathrm{e}^{q^{2+2b-\alpha_\tau} n^{-1/2+\alpha_\tau}} \\
&\qquad + q^{\frac{17}{4}+5b+(1+20b)\alpha_\tau} n^{-\frac{1}{2}-2\alpha_\tau} \mathrm{e}^{q^{2+(1/4+5b)\alpha_\tau} n^{-\alpha_\tau/2}}.
\end{aligned}
$$

Notice that we require $3/2 - 4\alpha_\tau > 0$, which is covered by our explicit choice of α_τ below.

The latter expression is bounded by a constant if

$$
\varepsilon < \varepsilon^\tau(\alpha_\tau, b) := \min\{\varepsilon_i^\tau(\alpha_\tau, b) : \ 1 \le i \le 4\},
$$

where

$$
\begin{aligned}
\varepsilon_1^\tau(\alpha_\tau, b) &= \frac{4 - 8\alpha_\tau}{7 + 16b - 8\alpha_\tau}, & \varepsilon_2^\tau(\alpha_\tau, b) &= \frac{1 - 2\alpha_\tau}{4 + 4b - 2\alpha_\tau}, \\
\varepsilon_3^\tau(\alpha_\tau, b) &= \frac{2 + 8\alpha_\tau}{17 + 20b + (4 + 80b)\alpha_\tau}, & \varepsilon_4^\tau(\alpha_\tau, b) &= \frac{2\alpha_\tau}{8 + (1 + 20b)\alpha_\tau}.
\end{aligned}
$$

It can easily be checked that $\varepsilon_1^\tau \ge \varepsilon_2^\tau$ and that $\varepsilon_3^\tau \ge \varepsilon_4^\tau$ for all $\alpha_\tau \in (0, 1/2)$. Additionally, ε_2^τ and ε_4^τ intersect along the curve

$$
\alpha_\tau = \frac{-23 + 12b + \sqrt{(3+4b)(155+36b)}}{80b - 4} =: \alpha_\tau^{\text{opt}}(b),
$$

where we have

$$
\varepsilon^\tau(\alpha_\tau^{\text{opt}}(b), b) = \frac{4}{25 + 28b + \sqrt{(3+4b)(155+36b)}} = \varepsilon^\tau(b)
$$

and, consequently, (2.59) follows.

It remains to show that the part of (2.54) containing Σ_{cycle} can be bounded by a constant as well. This can be derived for various choices of certain parameters made below. Recall that $b < 1/4$ is already dictated by Lemma 2.31, (2.26), and we therefore strive to achieve a constant upper bound valid for all $\varepsilon \leq \varepsilon^{\tau}(1/4)$.

It is immediately clear that bicolored cycles have to comprise at least four vertices. Therefore, we may estimate

$$\Sigma_{\text{cycle}} \lesssim \sum_{t=1}^{v/4} \sum_{\substack{t_1,\dots,t_l \geq 0 \\ t_1+\dots+t_l=t}} \sum_{\substack{G_1 \text{ a.c. on } V_1 \\ T(G_1)=t_1}} \cdots \sum_{\substack{G_l \text{ a.c. on } V_l \\ T(G_l)=t_l}} \prod_{j=1}^{l} \tilde{\rho}^{|V_j|} \left\| \mathrm{SP}\left(\mathbb{X}(G_j)\right)\right\|_{q^{2b}}$$
$$\lesssim (S_{<l} + S_{\geq l}) \prod_{j=1}^{l} M_{|V_j|,l}, \quad (2.66)$$

where we applied Lemma 2.42 with $\varepsilon < 2/(5+4b)$ and where

$$S_{<l} = \sum_{t=1}^{l-1} l^{-\frac{t}{2}} q^{\frac{t(1-2b)}{2}} n^{-\frac{t}{2}} \sum_{\substack{t_1,\dots,t_l \geq 0 \\ t_1+\dots+t_l=t}} \sum_{\substack{G_1 \text{ a.c. on } V_1 \\ T(G_1)=t_1}} \cdots \sum_{\substack{G_l \text{ a.c. on } V_l \\ T(G_l)=t_l}} 1 \quad \text{and}$$
$$S_{\geq l} = \sum_{t=l}^{v/4} l^{-\frac{t}{2}} q^{\frac{t(1-2b)}{2}} n^{-\frac{t}{2}} \sum_{\substack{t_1,\dots,t_l \geq 0 \\ t_1+\dots+t_l=t}} \sum_{\substack{G_1 \text{ a.c. on } V_1 \\ T(G_1)=t_1}} \cdots \sum_{\substack{G_l \text{ a.c. on } V_l \\ T(G_l)=t_l}} 1.$$

Let us take care of $S_{<l}$ first. Within the subsequent paragraphs we show

$$\sum_{l=1}^{v/2} \sum_{(V_1,\dots,V_l)\in\mathcal{V}(V,l)} \left(\prod_{j=1}^{l} M_{|V_j|,l}\right) S_{<l} \lesssim v^{(\frac{3}{2}-\alpha_{\tau}^{\text{opt}}(b))v-\frac{1}{2}} q^{(2b-\frac{1}{2})v} n^{-(\frac{1}{2}+\alpha_{\tau}^{\text{opt}}(b))v}, \quad (2.67)$$

i.e., the part containing $S_{<l}$ is subject to the upper bound of Σ_1^{τ} from (2.62). To this end, let us consider an admissible graph G with connected components $G_1,\dots,G_l$. Since $t < l$, at most t indices $j_1,\dots,j_t$ satisfy $T(G_{j_\tau}) \geq 1$. Thus, at least $l-t$ of those subgraphs do not contain a cycle and we may therefore estimate

$$S_{<l} \lesssim \sum_{t=1}^{l-1} l^{-\frac{t}{2}} q^{\frac{t(1-2b)}{2}} n^{-\frac{t}{2}} \binom{t+l-1}{l-1} \left(\frac{v}{l-t}\right)^{v-2(l-t)}$$
$$\lesssim \sum_{t=1}^{l-1} l^{-\frac{t}{2}} q^{\frac{t(1-2b)}{2}} n^{-\frac{t}{2}} (t+l)^{l-1} \frac{l}{l!} \left(\frac{v}{l-t}\right)^{v-2(l-t)}$$
$$\simeq l^{-\frac{1}{2}} v^{v-2l} \sum_{t=1}^{l-1} (l-t)^{-v+2(l-t)} \left(l^{-\frac{1}{2}} v^2 q^{\frac{1-2b}{2}} n^{-\frac{1}{2}}\right)^t .$$

Here, we used Lemma 2.45 with $k = l - t$ as well as $l \simeq t + l$ and Stirling's formula. For $\varepsilon < 1/(14-2b)$ we have $n^{1/2} \gtrsim q^{(1-2b)/2} v^{13/2}$ and , consequently,

$$S_{<l} \lesssim l^{-\frac{1}{2}} v^{v-2l} \sum_{t=1}^{l-1} (l-t)^{-v+2(l-t)} \left(l^{-\frac{1}{2}} v^{-\frac{9}{2}}\right)^t =: l^{-\frac{1}{2}} v^{v-2l} \sum_{t=1}^{l-1} H(t).$$

It needs to be added that $\varepsilon^\tau(b) \leq 1/(14-2b)$ for $b \geq 0.18\ldots$ The critical point t_0 of H on $[1, l-1]$ is given by the relation

$$l - t_0 = \frac{v}{2w}, \qquad \text{where } w = W\left(\frac{\mathrm{e}}{2} l^{\frac{1}{4}} v^{\frac{13}{4}}\right).$$

Recall that $t \geq 1$ implies $v \geq 4$, as has been observed earlier. Therefore, $\mathrm{e}2^{-1} l^{1/4} v^{\frac{13}{4}} \geq 2^{11/2}\mathrm{e}$, which in turn implies $w > 88/25$ and, thus,

$$-v + \frac{v}{w} \leq -\frac{63}{88} v.$$

On the other hand, for each $\kappa > 0$ there exists a constant $c > 0$ such that

$$w \leq \left(\frac{\mathrm{e}}{2} l^{\frac{1}{4}} v^{\frac{13}{4}}\right)^\kappa + c.$$

Therefore,

$$\frac{v}{2w} \gtrsim l^{-\frac{\kappa}{4}} v^{1-\frac{13\kappa}{4}}.$$

Observe that the latter expression is greater than one for $\kappa > 2/7$. Hence, we may estimate as follows

$$H(t_0) = \left(\frac{v}{2w}\right)^{-v+\frac{v}{w}} \left(l^{-\frac{1}{2}} v^{-\frac{9}{2}}\right)^{l-\frac{v}{2w}}$$
$$\lesssim \left(l^{-\frac{\kappa}{4}} v^{1-\frac{13\kappa}{4}}\right)^{-\frac{63}{88}v} \left(l^{-\frac{1}{2}} v^{-\frac{9}{2}}\right)^{l-\frac{25}{176}v} \simeq l^{\frac{25+63\kappa}{352}v - \frac{l}{2}} v^{-\frac{27-819\kappa}{352}v - \frac{9}{2}l}$$

At the boundaries the function $H(t)$ evaluates to

$$H(1) \simeq l^{-v+2l-\frac{5}{2}} v^{-\frac{9}{2}}, \qquad H(l-1) = l^{-\frac{l}{2}+\frac{1}{2}} v^{-\frac{9}{2}l+\frac{9}{2}}.$$

Choosing $\kappa = 1/441$, for instance, we immediately see that

$$25 + 63\kappa + 27 - 819\kappa = 0$$

and, consequently,

$$\frac{H(l-1)}{H(t_0)} \gtrsim 1.$$

Moreover,

$$H(1) \lesssim l^{-v+2l-\frac{1}{2}},$$

which yields the corresponding part in the estimation of Σ_{tree}, see (2.60). Therefore, we need only consider $H(l-1)$.

We proceed similarly as in the lines leading to (2.61), i.e.

$$\begin{aligned}
\sum_{l=1}^{\alpha_\tau^{\mathrm{opt}} v} & \sum_{(V_1,\dots,V_l)\in\mathcal{V}(V,l)} \left(\prod_{j=1}^{l} M_{|V_j|,l}\right) S_{<l} \\
&\lesssim \sum_{l=1}^{\alpha_\tau^{\mathrm{opt}} v} \binom{v}{l} l^{\frac{3}{2}v-l+\frac{1}{2}} v^{v-2l} q^{(2b-\frac{1}{2})v} n^{-\frac{v}{2}+l} H(l-1) \\
&\lesssim \sum_{l=1}^{\alpha_\tau^{\mathrm{opt}} v} l^{\frac{3}{2}v-\frac{5}{2}l+\frac{1}{2}} v^{v-\frac{11}{2}l+\frac{9}{2}} q^{(2b-\frac{1}{2})v} n^{-\frac{v}{2}+l} \\
&\lesssim \sum_{l=1}^{\alpha_\tau^{\mathrm{opt}} v} v^{\frac{5}{2}v-8l+5} q^{(2b-\frac{1}{2})v} n^{-\frac{v}{2}+l} \\
&\lesssim v^{(\frac{5}{2}-8\alpha_\tau^{\mathrm{opt}})v+5} q^{(2b-\frac{1}{2})v} n^{(-\frac{1}{2}+\alpha_\tau^{\mathrm{opt}})v},
\end{aligned}$$

where we used $\varepsilon < 1/8$ in the last step. Notice that the exponents of q and n already match those of the right-hand side of (2.67). Moreover, we observe that

$$\begin{aligned}
v\left(\frac{5}{2} - 8\alpha_\tau^{\mathrm{opt}}(b)\right) + 5 &= v\left(\frac{3}{2} - \alpha_\tau^{\mathrm{opt}}(b)\right) - \frac{1}{2} + v\left(1 - 7\alpha_\tau^{\mathrm{opt}}(b)\right) + \frac{11}{2} \\
&\le \left(\frac{3}{2} - \alpha_\tau^{\mathrm{opt}}(b)\right) - \frac{1}{2} + \frac{19}{2} - 28\alpha_\tau^{\mathrm{opt}}(b) < \left(\frac{3}{2} - \alpha_\tau^{\mathrm{opt}}(b)\right) - \frac{1}{2}.
\end{aligned}$$

The latter expression, in turn, appears as the claimed exponent of v and the result follows for $l \le \alpha_\tau^{\mathrm{opt}}(b)v$.

Finally, let $l \geq \alpha_\tau^{\mathrm{opt}}(b)v + 1$. In this case we have

$$\frac{H(l-1)}{H(1)} \simeq l^{v-\frac{5}{2}l+3} v^{-\frac{9}{2}l+9} \lesssim l^{(1-\frac{5\alpha_\tau^{\mathrm{opt}}(b)}{2})v+\frac{1}{2}} v^{-\frac{9\alpha_\tau^{\mathrm{opt}}(b)}{2}v+\frac{9}{2}} \lesssim v^{v(1-7\alpha_\tau^{\mathrm{opt}}(b))+5} \lesssim v^{9-28\alpha_\tau^{\mathrm{opt}}(b)} \lesssim 1$$

and we are done proving (2.67).

Let us now draw our attention to the part containing $S_{\geq l}$. As a consequence of Lemma 2.39 and Lemma 2.45 we obtain without difficulty

$$\begin{aligned} S_{\geq l} &\lesssim \sum_{t=l}^{v/4} l^{-\frac{t}{2}} q^{\frac{1-2b}{2}t} n^{-\frac{t}{2}} \binom{t+l-1}{l-1} \left(\frac{v}{l}\right)^{2v} \\ &\lesssim l^{-2v-l+\frac{1}{2}} v^{2v} \sum_{t=l}^{v/4} l^{-\frac{t}{2}} (t+l-1)^{l-1} q^{\frac{1-2b}{2}t} n^{-\frac{t}{2}} \\ &\lesssim l^{-2v-l+\frac{1}{2}} v^{2v+l-1} \sum_{t=l}^{v/4} \left(l^{-\frac{1}{2}} q^{\frac{1-2b}{2}} n^{-\frac{1}{2}}\right)^t \\ &\lesssim l^{-2v-\frac{3}{2}l+\frac{1}{2}} v^{2v+l-1} q^{\frac{1-2b}{2}l} n^{-\frac{l}{2}}, \end{aligned}$$

since, clearly, $\varepsilon < 1/(1-2b)$. Subsequently, we proceed analogously to (2.61) to find

$$\begin{aligned} &\sum_{l=1}^{v/2} \sum_{(V_1,\ldots,V_l)\in\mathcal{V}(V,l)} \left(\prod_{j=1}^{l} M_{|V_j|,l}\right) S_{\geq l} \\ &\quad\lesssim \sum_{l=1}^{\alpha_\tau^{\mathrm{opt}}(b)v} \binom{v}{l} l^{-\frac{v}{2}-\frac{5}{2}l+\frac{1}{2}} v^{2v+l-1} q^{\frac{4b-1}{2}v+\frac{1-2b}{2}l} n^{-\frac{v}{2}+\frac{l}{2}} \\ &\qquad + \sum_{l=\alpha_\tau^{\mathrm{opt}}(b)v+1}^{v/2} \binom{v}{l} l^{-v-l+\frac{1}{2}} v^{2v+l-1} q^{(\frac{1}{4}+4b)l} n^{-l} \\ &\quad=: \Sigma_1^\kappa + \Sigma_2^\kappa, \end{aligned}$$

where we used Lemma 2.44. In what follows, we show that Σ_1^κ and Σ_2^κ are subject to the same upper bounds as Σ_1^τ and Σ_2^τ, respectively, see (2.62) and

(2.65). Indeed,

$$
\begin{aligned}
\Sigma_1^\kappa &\lesssim \sum_{l=1}^{\alpha_\tau^{\mathrm{opt}}(b)v} l^{-\frac{v}{2}-\frac{7}{2}l} v^{2v+2l-1} q^{\frac{4b-1}{2}v+\frac{1-2b}{2}l} n^{-\frac{v}{2}+\frac{l}{2}} \\
&\lesssim v^{(2+2\alpha_\tau^{\mathrm{opt}}(b))v-1} q^{(\frac{4b-1}{2}+\frac{(1-2b)\alpha_\tau^{\mathrm{opt}}(b)}{2})v} n^{-\frac{1-\alpha_\tau^{\mathrm{opt}}(b)}{2}v} \\
&\le v^{(\frac{3}{2}-\alpha_\tau^{\mathrm{opt}}(b))v} q^{\frac{4b-1}{2}v} n^{-(\frac{1}{2}-\alpha_\tau^{\mathrm{opt}}(b))v} v^{(\frac{1}{2}+3\alpha_\tau^{\mathrm{opt}}(b))v} q^{\frac{(1-2b)\alpha_\tau^{\mathrm{opt}}(b)}{2}v} n^{-\frac{\alpha_\tau^{\mathrm{opt}}(b)}{2}v} \\
&\lesssim v^{(\frac{3}{2}-\alpha_\tau^{\mathrm{opt}}(b))v} q^{\frac{4b-1}{2}v} n^{-(\frac{1}{2}-\alpha_\tau^{\mathrm{opt}}(b))v},
\end{aligned}
$$

where we used

$$
\frac{\alpha_\tau^{\mathrm{opt}}(b)}{1+(7-2b)\alpha_\tau^{\mathrm{opt}}(b)} = \frac{16}{135-44b+\sqrt{(3+4b)(155+36b)}} > \varepsilon^\tau(b)
$$

in the last step.

For the second sum, Σ_2^κ, we obtain by (2.63)

$$
\begin{aligned}
\Sigma_2^\kappa = &\sum_{l=0}^{v/2-\alpha_\tau^{\mathrm{opt}}(b)v-1} \binom{v}{l+\alpha_\tau^{\mathrm{opt}}(b)v+1} (l+\alpha_\tau^{\mathrm{opt}}(b)v+1)^{-(1+\alpha_\tau^{\mathrm{opt}}(b))v-l-\frac{1}{2}} v^{(2+\alpha_\tau^{\mathrm{opt}}(b))v+l} \\
&\qquad \times q^{(\frac{\alpha_\tau^{\mathrm{opt}}(b)}{4}+4b\alpha_\tau^{\mathrm{opt}}(b))v+(\frac{1}{4}+4b)l+\frac{1}{4}+4b} n^{-\alpha_\tau^{\mathrm{opt}}(b)v-l-1} \\
\lesssim &\sum_{l=0}^{v-\alpha_\tau^{\mathrm{opt}}(b)v-1} \binom{v-\alpha_\tau^{\mathrm{opt}}(b)v-1}{l} (l+\alpha_\tau^{\mathrm{opt}}(b)v+1)^{-(1+2\alpha_\tau^{\mathrm{opt}}(b))v-l-\frac{3}{2}} v^{(2+2\alpha_\tau^{\mathrm{opt}}(b))v+l+1} \\
&\qquad \times q^{(\frac{\alpha_\tau^{\mathrm{opt}}(b)}{4}+4b\alpha_\tau^{\mathrm{opt}}(b))v+(\frac{1}{4}+4b)l+\frac{1}{4}+4b} n^{-\alpha_\tau^{\mathrm{opt}}(b)v-l-1} \\
\simeq &\; v^{v-\frac{1}{2}} q^{(\frac{\alpha_\tau^{\mathrm{opt}}(b)}{4}+4b\alpha_\tau^{\mathrm{opt}}(b))v+\frac{1}{4}+4b} n^{-\alpha_\tau^{\mathrm{opt}}(b)v-1} \sum_{l=0}^{v-\alpha_\tau^{\mathrm{opt}}(b)v-1} \binom{v-\alpha_\tau^{\mathrm{opt}}(b)v-1}{l} \left(q^{\frac{1}{4}+4b} n^{-1}\right)^l \\
= &\; v^{v-\frac{1}{2}} q^{(\frac{\alpha_\tau^{\mathrm{opt}}(b)}{4}+4b\alpha_\tau^{\mathrm{opt}}(b))v+\frac{1}{4}+4b} n^{-\alpha_\tau^{\mathrm{opt}}(b)v-1} \left(1+q^{\frac{1}{4}+4b} n^{-1}\right)^{v-\alpha_\tau^{\mathrm{opt}}(b)-1} \\
\lesssim &\; v^{v-\frac{1}{2}} q^{(\frac{\alpha_\tau^{\mathrm{opt}}(b)}{4}+4b\alpha_\tau^{\mathrm{opt}}(b))v+\frac{1}{4}+4b} n^{-\alpha_\tau^{\mathrm{opt}}(b)v-1} \mathrm{e}^{vq^{1/4+4b}n^{-1}} \\
\lesssim &\; v^{v-\frac{1}{2}} q^{(\frac{\alpha_\tau^{\mathrm{opt}}(b)}{4}+4b\alpha_\tau^{\mathrm{opt}}(b))v+\frac{1}{4}+4b} n^{-\alpha_\tau^{\mathrm{opt}}(b)v-1}
\end{aligned}
$$

for $\varepsilon < 4/(5+16b)$. Observe that $\varepsilon^\tau(b)$ is dominated by this expression. We may now continue as follows

$$
\begin{aligned}
&v^{v-\frac{1}{2}} q^{(\frac{\alpha_\tau^{\mathrm{opt}}(b)}{4}+4b\alpha_\tau^{\mathrm{opt}}(b))v+\frac{1}{4}+4b} n^{-\alpha_\tau^{\mathrm{opt}}(b)v-1} \\
&\qquad \lesssim v^{(1+\frac{\alpha_\tau^{\mathrm{opt}}(b)}{2})v+\frac{1}{2}} q^{(5b\alpha_\tau^{\mathrm{opt}}(b)-\frac{\alpha_\tau^{\mathrm{opt}}(b)}{4})v+5b-\frac{1}{4}} q^{\frac{\alpha_\tau^{\mathrm{opt}}(b)}{2}v+\frac{1}{2}} n^{-\alpha_\tau^{\mathrm{opt}}(b)v-1} \\
&\qquad\qquad \lesssim v^{(1+\frac{\alpha_\tau^{\mathrm{opt}}(b)}{2})v+\frac{1}{2}} q^{(5b\alpha_\tau^{\mathrm{opt}}(b)-\frac{\alpha_\tau^{\mathrm{opt}}(b)}{4})v+5b-\frac{1}{4}} n^{-\frac{\alpha_\tau^{\mathrm{opt}}(b)}{2}v-\frac{1}{2}},
\end{aligned}
$$

which is the upper bound for Σ_2^τ (2.65). This finally concludes the proof of (2.27) and thus of Lemma 2.31, too.

2.3.7 The lower bound for the inner product

Until now, we have merely focused on finding an upper bound for the inner product $\langle D_N(\mathcal{P},\cdot),\Psi^{\mathrm{sd}}\rangle$. In this section we put our emphasis on retreiving a lower bound. More precisely, we show the following lemma below.

Lemma 2.46. *There exist* r*-functions such that the inner product of the discrepancy function with the strongly distinct part of* Ψ *satisfies*

$$\left\langle D_N(\mathcal{P},\cdot),\Psi^{sd}\right\rangle \gtrsim aq^b n \simeq n^{1+\varepsilon/4}.$$

To prove this, we follow the approach of Halász, or its refinement by Bilyk, Lacey, and Vagharshakyan to be more precise, see [8, 13] for instance. Recall that Ψ^{sd} admits of a decomposition of the form

$$\Psi^{\mathrm{sd}} = \Psi_1^{\mathrm{sd}} + \Psi_2^{\mathrm{sd}} + \cdots + \Psi_q^{\mathrm{sd}},$$

see (2.25). Moreover, notice that Ψ is only defined up to a selection of signs within the employed r-functions. In the lemma below we confine ourselves to a fixed choice of signs for which we can provide a good lower bound of $\langle D_N(\mathcal{P},\cdot),\Psi_1^{\mathrm{sd}}\rangle$, see Lemma 2.47. Subsequently, we show that all other summands contribute significantly less to the total result, see Lemma 2.48.

Lemma 2.47 (Cf. [13, p. 2500])**.** *There exist* r*-functions such that*

$$\langle D_N(\mathcal{P},\cdot),\Psi_1^{sd}\rangle \gtrsim q^b n.$$

Proof. We choose our r-functions $f_{\vec{r}}$, $\vec{r}\in\mathbb{H}_n^3$, as we did in (2.15) and obtain

$$\langle D_N(\mathcal{P},\cdot),f_{\vec{r}}\rangle \gtrsim 1$$

by Lemma 2.28. Inserting the definition of Ψ_1^{sd} this immediately yields

$$\left\langle D_N(\mathcal{P},\cdot),\Psi_1^{\mathrm{sd}}\right\rangle = \sum_{v=1}^{q}\sum_{\vec{r}\in\mathbb{A}_v}\tilde{\rho}\,\langle D_N(\mathcal{P},\cdot),f_{\vec{r}}\rangle \gtrsim q\cdot n^2 q^{-1}\cdot\tilde{\rho}\simeq q^b n.$$

□

For the higher order terms we can show the following analogon to certain steps in Halász' proof.

Lemma 2.48 (Cf. [13, p. 2500])**.** *The higher order terms of* Ψ^{sd}*, i.e.* Ψ^{sd}_v*, $2 \le v \le q$ as defined in (2.25), are subject to*

$$\sum_{v=2}^{q} \left|\left\langle D_N(\mathcal{P},\cdot), \Psi^{sd}_v \right\rangle\right| \lesssim 1.$$

Proof. As a direct consequence of the product rule (Proposition 2.25) and item (iv) of Remark 2.27 each Ψ^{sd}_v, $v \ge 2$, is a sum of r-functions with certain parameters $\vec{s}$. Moreover, the range of values that the first coordinates of the individual hyperbolic vectors which define $\vec{s}$ can take on, stretches over v sets I_j of our partition (2.21). Hence, their maximum is at least $(v-1)n/q+1$ and it is certainly not attained in I_1. Therefore,

$$n + \frac{(v-1)n}{q} \le \|\vec{s}\|_{\ell^1} \le 3n.$$

By one application of the triangle inequality we obtain

$$\left|\left\langle D_N(\mathcal{P},\cdot), \Psi^{\mathrm{sd}}_v \right\rangle\right| \le \tilde{\rho}^v \sum_{h=(v-1)n/q}^{2n} \sum_{\|\vec{s}\|_{\ell^1}=n+h} |\langle D_N(\mathcal{P},\cdot), f_{\vec{s}}\rangle| \, \mathrm{count}(\vec{s},v),$$

where $\mathrm{count}(\vec{s},v)$ is defined as

$$\#\left\{(\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_v) : \ \vec{r}_j \in \mathbb{H}^3_n \text{ and } \prod_{j=1}^{v} f_{\vec{r}_j} \text{ is an } \mathsf{r}\text{-function with parameter } \vec{s}\right\}.$$

This entity can be crudely estimated by

$$\mathrm{count}(\vec{s},v) \le \left(\#\mathbb{H}^3_n\right)^v = \binom{\|\vec{s}\|_{\ell^1}+2}{2}^v \lesssim \|\vec{s}\|^{2k}_{\ell^1}.$$

This together with Lemma 2.29 implies

$$\left|\left\langle D_N(\mathcal{P},\cdot), \Psi^{\mathrm{sd}}_v \right\rangle\right| \lesssim \left(\frac{q^b}{n}\right)^v \sum_{h=(v-1)n/q}^{2n} \#\mathbb{H}^3_{n+h} \cdot N 2^{-n-h}(n+h)^{2v}.$$

Since $n \simeq \log_2 N$ and $\#\mathbb{H}^3_{n+h} \lesssim (n+h)^2$ we obtain further

$$\left|\left\langle D_N(\mathcal{P},\cdot), \Psi^{\mathrm{sd}}_v \right\rangle\right| \lesssim \left(\frac{q^b}{n}\right)^v \sum_{h=(v-1)n/q}^{2n} 2^{-h}(n+h)^{2v+2} \lesssim \left(\frac{q^b}{n}\right)^v 2^{-\frac{(v-1)n}{q}}(3n)^{2v+3}.$$

Noticing that $2^{-n(v-1)/q} \leq 2^{-nv/(2q)}$ as well as that the summand for $v = 2$ is of negligible size we may continue for $v \geq 3$ by

$$\left|\left\langle D_N(\mathcal{P}, \cdot), \Psi_v^{\mathrm{sd}} \right\rangle\right| \lesssim \left(\frac{q^b}{n}\right)^v 2^{-\frac{nv}{2q}} (3n)^{3v} = \left(\frac{27 q^b n^2}{2^{\frac{n}{2q}}}\right)^v,$$

which adds up to a constant for n sufficiently large. □

The verification of Lemma 2.46 is a particularly easy task now.

Proof of Lemma 2.46. Due to Lemma 2.47 and Lemma 2.48 we immediately see

$$\left\langle D_N(\mathcal{P}, \cdot), \Psi^{\mathrm{sd}} \right\rangle \gtrsim \left\langle D_N(\mathcal{P}, \cdot), \Psi_1^{\mathrm{sd}} \right\rangle - \sum_{v=2}^{q} \left|\left\langle D_N(\mathcal{P}, \cdot), \Psi_v^{\mathrm{sd}} \right\rangle\right| \gtrsim q^b n.$$

□

2.3.8 Discussion and open problems

Theorem 2.23 is valid in three dimensions only. The proof of the existence result Theorem 2.5 in $d \geq 4$, which is given in [13], differs from the three-dimensional approach. It would therefore be interesting to know which constant η_d can be extracted from this proof, or, whether it is possible to apply similar strategies as the ones presented here and in [69].

Open Problem 2.49. Does [13] allow to find a good explicit constant η_d in Theorem 2.5? Furthermore, is the approach from this thesis to determine a value for η_3 in some way extendable to dimensions $d \geq 4$?

As it has already been mentioned earlier, the papers [11, 13] mainly emphasize on a different problem, i.e. the SBI and include the star discrepancy estimate as a further result. But the methods of proof are strikingly similar, i.e. they still use (2.29), but with a different function on the dual side of Ψ^{sd}. Additionally, a connection between these two subjects is formally established for $d = 2$ in [9]. As a matter of fact, the link was drawn to the SSBI, a special instance of the SBI. Moreover, in [10] a considerably simpler argument completely obsoleted the tedious study of coincidences of length greater than 2 in the SSBI, while the overall strategy was maintained.

Open Problem 2.50. Is the argument from [10] transferable to the proof of lower star discrepancy bounds, thus avoiding the study of long coincidences and maybe simultaneously answering Open Problem 2.49.

While the optimal growth rate of the L^p-norm of the discreancy function is known to be $(\log N)^{(d-1)/2}$ for $1 < p < \infty$ (see Theorem 2.1 and [14, 76]), the rate of the star discrepancy remains a subject of speculation. Even the conjectures are disputed among experts. In order to get an idea what might happen at the transition from $p < \infty$ to $p = \infty$ one might look at a space that is considered close to L^∞, i.e. the so-called exponential Orlicz spaces. For $\beta \geq 1$ and $\psi(x) = \mathrm{e}^{x^\beta} - 1$ the exponential Orlicz space $\exp(L^\beta)$ is given by all functions satisfying

$$\|f\|_{\exp(L^\beta)} = \inf \left\{ K > 0 : \mathbb{E}\psi\left(\frac{|f|}{K}\right) \leq 1 \right\}.$$

The open problem below was brought to my attention by Michael Lacey, since it might be eventually proved by arguments related to those used in [8, 11, 69] and via the equivalence of norms

$$\|f\|_{\exp(L^\beta)} \simeq \sup_{p>1} p^{-\beta} \|f\|_p.$$

We state the problem in its full generality, although any contribution into this direction would already be extremely useful. The $d = 2$ case of the conjecture below has already been shown in [12].

Open Problem 2.51. For all $d \geq 2$ and all N-point sets $\mathcal{P}$ we have

$$\|D_N^*(\mathcal{P}, \cdot)\|_{\exp(L^\beta)} \gtrsim (\log N)^{\frac{d}{2} - \frac{1}{\beta}}, \qquad 2 \leq \beta < \infty.$$

The exponent $d/2$ may as well be exchanged by any of the analogons from Conjectures 2.3 or 2.7.

Chapter 3

Explicit examples for classical and hybrid sequences

Let us now put our emphasis on the second main field of interest regarding discrepancy theory, i.e. finding explicit construction principles for envenly distributed point sets and sequences. Within this thesis we only pursue the latter, however, many classes of *good* point sets are known. We refer the reader to the books [18, 61] for several examples.

It needs to be added that, up to now, the best constructions yield an upper bound for the star discrepancy of $(\log N)^d$, which serves as an incentive for the following definition.

Definition 3.1. A sequence $\mathcal{S}$ in the d-dimensional unit cube is called a low-discrepancy sequence if and only if its star discrepancy satisfies

$$D_N^*(\mathcal{S}) \lesssim (\log N)^d .$$

At first, we present two classical sequences in Section 3.1 which we have already encountered earlier in this thesis. The first one is the Kronecker sequence $(\{k\alpha\})_{k\geq 0}$ (cf. Definition 1.3) for which we already know that it is u.d. mod1 for irrational α (cf. Corollary 1.5). Our second example is of a more complex nature and originates from Niederreiter [60]. Sequences of this class are commonly referred to as *digital sequences.* As a special instance they contain the *Van der Corput sequence*, which we have mentioned to verify the sharpness of Schmidt's bound (Theorem 2.2) earlier in Section 2.1.

In the second part of this chapter we combine two one-dimensional copies of these sequences to form one two-dimensional sequence. One might see Spanier ([78]) as a precursor to this *hybrid* approach and we will discuss certain incentives for considering hybrid sequences in general in Section 3.2.1 before we turn to the distribution properties of our specific example in Section 3.2.2.

3.1 Two classical sequences

Before we proceed we need to introduce some further notation. In what follows we write $A(N) \ll_X B(N)$ if $|A(N)| \leq c_X |B(N)|$ for all N large enough and $A(N) \gg_X B(N)$ if $|A(N)| \geq c_X |B(N)|$ for infinitely many $N \in \mathbb{N}$, where $c_X > 0$ is a constant depending on a collection of parameters indicated by X. Moreover, we write $A(N) \asymp_X B(N)$ if $A(N) \ll_X B(N)$ and $A(N) \gg_X B(N)$. Notice the subtle difference to the symbols "$\lesssim$" and "$\gtrsim$" in terms of their validity w.r.t. N.

3.1.1 The Kronecker sequence

One of the most basic attempts of finding low-discrepancy sequences is taking the fractional part of multiples of a fixed irrational number α. I.e., we consider the so-called Kronecker sequence $(\{k\alpha\})_{k\geq 0}$, as we have already defined in Definition 1.3. Its distribution properties can be closely linked to the partial continued fraction expansion of α. In all its brevity, this means that we write α as

$$\alpha = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots}}}$$

with $a_0 \in \mathbb{Z}$ and positive integers a_j, $j \geq 1$. We abbreviate this expansion by $\alpha = [a_0; a_1, a_2, \ldots]$. Since we consider $\alpha \in (0,1)$ only, we automatically have $a_0 = 0$. Furthermore, it is commonly known that this expansion is finite, i.e. $a_j = 0$ for all $j \geq j_0$ sufficiently large, if and only if α is rational. Hence, in our setting, infinitely many of the *continued fraction coefficients* a_j are non-zero. Good bounds for the discrepancy of Kronecker sequences valid for all $\alpha \in (0,1) \setminus \mathbb{Q}$ are not available in general. However, narrowing down the choices for α the theorem below can be obtained. This was initially shown by Ostrowski [68] (1922), see also [61, Corollary 3.4].

Theorem 3.2. *Let* $\alpha = [0; a_1, a_2, \ldots]$ *with* $\sum_{j=1}^{m} a_j \leq cm$ *for all* m *and an absolute constant* $c > 0$. *Then*

$$D_N^* \left((\{k\alpha\})_{k\geq 0} \right) \ll \log N.$$

This result even holds for all $N \geq 2$.

The above shows, that the Kronecker sequence can be a low discrepancy sequence for certain choices of α. One of the immediate base cases to which this theorem applies are those parameters with bounded continued fraction coefficients (b.c.f.c.), i.e. $a_j \leq K$ for some natural number K and all j. A prominent example for such numbers are the quadratic irrational numbers, as they appear to be the only numbers with a periodic continued fraction expansion. For a survey on continued fractions see [40], for instance.

According to Schoissengeier ([74])also the converse of Theorem 3.2 is true. That is, if for all $N \geq 2$ we have $D_N^*\left((\{k\alpha\})_{k\geq 0}\right) \ll \log N$ then $\sum_{j=1}^m a_j \leq cm$. However, it is known that almost all α in the sense of the Lebesgue measure are subject to $\sum_{j=1}^m a_j > m\log m$ for m large enough (see [40]). Consequently, the set of all α for which the Kronecker sequence is a low-discrepancy sequence has measure zero. Nevertheless, *metric* discrepancy estimates reveal that the situation is brighter than one would expect. Here, the term *metric* refers to results valid for almost all α w.r.t. the Lebesgue measure.

Theorem 3.3. *For almost all* $\boldsymbol{\alpha} \in [0,1)^d$ *we have*

$$D_N^*\left((\{k\boldsymbol{\alpha}\})_{k\geq 0}\right) \ll_{\boldsymbol{\alpha},d} (\log N)^{d+\epsilon}$$

for all $\epsilon > 0$.

This means that for almost all $\boldsymbol{\alpha}$ we get arbitrarily close to the optimal (i.e., conjectured to be optimal) bound. The theorem above is due to Khintchine ([39]) in $d = 1$ relying on continued fraction expansions and to Beck ([5]) in all dimensions using a Poisson summation formula as well as probabilistic diophantine approximation.

Summarizing, we have seen that in one dimension we can explicitly name a class of parameters for which the optimal growth rate of the discrepancy is obtained although the class is comparably small. From another perspective, however, we know that in any dimension almost every parameter yields a highly favourable bound. It needs to be added that there is a good reason why Theorem 3.2 is formulated only for $d = 1$. Quite surprisingly, not even one pair of numbers (α_1, α_2) is known such that $(\{k\alpha_1\}, \{k\alpha_2\})_{k\geq 0}$ satisfies the upper bound from Theorem 3.3, although the probability for picking one at random is at 100 %.

3.1.2 Digital sequences

In this section we consider a relatively general class of sequences, the so-called (t,d)-sequences. As this subject alone (together with their finite analogons (t,m,d)-nets) fills entire books (see [18], for instance) we present a construction principle going back to Niederreiter [60] via a collection of d infinite matrices. For a base $b \geq 2$ and non-negative integers a_j, d_j, $a_j < b^{d_j}$ for $1 \leq j \leq d$, we define the following generalization of dyadic intervals

$$E = \prod_{j=1}^d \left[a_j b^{-d_j}, (a_j+1)b^{-d_j}\right)$$

and call them *elementary interval* in base b, cf. [60,61]. The main idea behind (t,d)-sequences $\mathcal{X} = (x_k)_{k\geq 0}$ is that every elementary interval in base b of volume b^{t-m} contains a fair share of points, i.e. exactly b^t elements within every finite segment $\{x_k : lb^m \leq k \leq (l+1)b^m\}$ of $\mathcal{X}$ for all $l \geq 0$ and $m > t$. In general, we have the following theorem, see e.g. [18,61].

Theorem 3.4. *Every (t,d)-sequence in base b satisfies*

$$D_N^*(\mathcal{X}) \ll_{b,d} b^t (\log N)^d$$

for all $N \geq 2$.

Consequently, $(0,d)$-sequences are of particular interest as they form low-discrepancy sequences.

The object we are going to describe more closely are so-called digital sequences. Although the construction scheme that is explained within this paragraph works in a more general setting as well (see e.g. [18]) we confine ourselves to b being prime. We choose a collection of d infinite matrices $C_1, C_2, \ldots, C_d$ over $\{0,1,\ldots,b-1\}$ and call them *generating matrices*. Furthermore, we expand each k into

$$k = k_0 + k_1 b + k_2 b^2 + \cdots, \qquad\qquad 0 \leq k_j < b$$

and compute

$$C_j \cdot (k_0, k_1, \ldots)^\top =: (y_j^{(1)}(k), y_j^{(2)}(k), \ldots)^\top.$$

Subsequently, the j-th coordinate of x_k, i.e. $x_k^{(j)}$, is constructed via

$$x_k^{(j)} = y_j^{(1)}(k) b^{-1} + y_j^{(2)}(k) b^{-2} + \cdots, \qquad 1 \leq j \leq d.$$

Finally, we set

$$x_k = \left(x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(d)}\right).$$

In the most basic setting we have $d = 1$ and $C_1 = \mathrm{Id}$, i.e. the identity matrix. In this case, the element x_k is obtained by reflecting the b-adic expansion of k at the comma position. This means, if $k = k_0 + k_1 b + \cdots$ then $x_k = k_0 b^{-1} + k_1 b^{-2} + \cdots$. Notice that this coincides with the definition of the Van der Corput sequence. By a slight generalization of the above construction scheme, i.e., allowing different bases $b_1, b_2, \ldots, b_d$ for each coordinate and taking $C_1 = \ldots = C_d = \mathrm{Id}$ again, we obtain the Halton sequence. Both of them, the Van der Corput sequence and its d-dimensional analogon, the Halton sequence, are known to be low-discreancy sequences, see [18,27,61].

As a matter of fact, this is the case for all integer bases $b_1, b_2, \dots, b_d$, $b_j \geq 2$, which are pairwise relatively prime.

It is known that the quality of digital sequences as described above depends on a certain linear independence parameter $\rho_m = \rho_m(C_1, \dots, C_d)$, which is defined as follows (cf. [18, Definition 4.82]). Let us denote by $C_j^{(m)}$ the upper left $m \times m$ submatrix of C_j. Then, the number ρ_m is the maximum of all non-negative integers r such that for any collection of integers $r_1, r_2, \dots, r_d \geq 0$, $r_1 + r_2 + \cdots + r_d = r$, with the property that

- the first r_1 rows of $C_1^{(m)}$ together with
- the first r_2 rows of $C_2^{(m)}$ together with
- ...
- the first r_d rows of $C_d^{(m)}$

are linearly independent over the field $\mathbb{Z}/b\mathbb{Z}$. The theorem below indicates that the digital sequences with the highest possible linear independence (in terms of ρ_m) form low-discrepancy sequences, and is a direct consequence of [18, Theorem 4.84].

Theorem 3.5. *Let $\mathcal{X}$ be the digital sequence in $[0,1)^d$ constructed by the generating matrices $C_1, C_2, \dots, C_d$ in prime base b. Then $\mathcal{X}$ is a $(0,d)$-sequence if*

$$\rho_m(C_1, C_2, \dots, C_d) = m$$

for all $m \geq 1$.

Notice that in one dimension it suffices to choose C_1 non-singular. Further famous examples of digital sequences are Sobol' sequences [77], Faure sequences [20], Niederreiter sequences [58], as well as Niederreiter-Xing sequences [63].

3.2 A hybrid approach

Here, we consider a combination of the two sequences that have been introduced in the previous section. More precisely, we take a one-dimensional Kronecker sequence as well as a one-dimensional digital sequence generated by a certain matrix C in base 2 and combine them to one two-dimensional sequence. In some sense, C is chosen *close* to the identity, i.e. the resulting sequence is *close* to the one-dimensional Halton- or Van der Corput sequence, hence the name *perturbed Halton–Kronecker sequence*. We set out this choice more clearly in Section 3.2.2.

We discuss the motivation of considering such *hybrid* constructions and give several examples in Section 3.2.1 below. Subsequently, Section 3.2.2 serves as an introduction to this type of sequences and presents the main results. These results are due to a joint work of Hofer together with the author ([37]), which has been submitted for publication. The two succeeding sections then cover the two different types of discrepancy bounds w.r.t. the Kronecker parameter α. I.e., in Section 3.2.3 we first derive a relatively general upper bound for the star discrepancy before we specialize towards α with b.c.f.c. In the latter case we also obtain a sharp lower bound. Whereas in Section 3.2.4 we focus on both upper and lower metric discrepancy bounds. Finally, Section 3.2.5 contains a thorough study of lacunary trigonometric products, as sharp general and tight metric estimates for these appear to be inevitable for our discrepancy estimates. In the end, we state several open problems that arise from our studies in Section 3.2.6.

3.2.1 A preliminary note on hybrid sequences

As it has already been discussed in the beginning of this thesis (see Section 1.2), numerical integration by QMC algorithms suffers from the curse of dimensionality in the genereal setting, while Monte Carlo algorithms do not. However, this comes at the cost of a significantly worse (expected) convergence rate in terms of the number of integration nodes N. In [78] Spanier came up with the idea of combining these approaches to exploit the benefits of both. He proposed to employ deterministic point sets for influential or significant coordinates and to use random nodes for the less important variables, thus making use of the good convergence rate of QMC while keeping the effective dimension at a necessary minimum. The approach of combining deterministic sequences with (pseudo) random numbers was followed in [55, 56, 62], just to name a few.

With a view to pure QMC methods, one might take Spanier's ideas one step further and come to the conclusion that different coordinates of the integrand might be best suited for different types of sequences. For instance, this could happen if various projections of the integrand belong to different function spaces. It could thus be benificial to combine two or more (classical) deterministic sequences in such cases.

Apart from that, one might stumble upon a new source of low-discrepancy sequences by juxtaposing two or more classical sequences. This attempt has gained much popularity recently, leading to a vast source of literature. See [28, 32–34, 36, 42, 43] for several examples.

Additionally, quite often the study of hybrid sequences leads to interesting number theoretic problems. For instance, the study of Halton–Kronecker sequences gave rise to the question of a p-adic generalization of the Thue–Siegel–Roth theorem, wich was solved by Ridout in [71]. Another example that will occur later in this thesis, is the need of irrational numbers with b.c.f.c. whose dyadic expansion is explicitely known. One such number was discovered by Shallit in [75].

3.2.2 Perturbed Halton–Kronecker sequences

The last paragraph of the previous section has already mentioned a precursor of the objects in which we are interested here, namely the so-called Halton–Kronecker sequences. They serve as our first explicit example where we combine two (almost) low-discrepancy sequences (see Theorem 3.2, Theorem 3.3, and Theorem 3.5) from completely different classes to one hybrid sequence. As a generic example, this combination has grown particularly famous and has been thoroughly studied over the recent years, see e.g. [19, 35, 47, 57, 59].

Let us now specify which sequence we are going to investigate.

Definition 3.6. We consider a digital sequence $(x_k(\ell))_{k\geq 0}$ in base 2 whose generating matrix C is the identity with its first row perturbed by a sequence $\boldsymbol{c}$ with period ℓ of the form

$$\boldsymbol{c} = (\underbrace{10\ldots0}_{\ell}\underbrace{10\ldots0}_{\ell}\ldots), \tag{3.1}$$

i.e.

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

where $c_j = 1$ for all j which are multiples of ℓ, and $c_j = 0$ in any other case. We call $\boldsymbol{c}$ a *perturbing sequence* and we refer to $(x_k(\ell))_{k\geq 0}$ as a perturbed Halton sequence.

Finally, the two-dimensional hybrid sequence we are interested in is given by $(\boldsymbol{z}_k(\ell))_{k\geq 0}$, where $\boldsymbol{z}_k(\ell) = (x_k(\ell), \{k\alpha\})$. We call this sequence a *perturbed Halton–Kronecker sequence.*

The first obvious extreme case are the Halton–Kronecker sequences which correspond to $(\boldsymbol{z}_k(\infty))_{k\geq 0}$. We have already seen that the behavior of the individual sequences is (supposedly) optimal if the parameter α has b.c.f.c. Unfortunately, this is not the case in the hybrid setting, see Theorem 2 and the subsequent paragraph of [19].

Theorem 3.7. *If the irrational number $\alpha \in [0,1)$ has b.c.f.c then the star discrepancy of the Halton–Kronecker sequence satisfies*

$$D_N^*\left((\boldsymbol{z}_k(\infty))_{k\geq 0}\right) \ll N^{\frac{1}{2}} (\log N)^{\frac{1}{2}}.$$

Moreover, for $\alpha = \sum_{l\geq 0} 4^{-2^l}$, which has b.c.f.c (cf. [75]), the corresponding sequence is subject to

$$D_N^*\left((\boldsymbol{z}_k(\infty))_{k\geq 0}\right) \gg N^{\frac{1}{2}}.$$

It immediately leaps to the eye that we may not even count on obtaining the optimal order of N, let alone of the logarithmic term. Hence, we put our focus on the exponent of N and take care of the $\log N$ part by adding or substracting an arbitrarily small $\epsilon > 0$ to/from this exponent.

The situation looks quite different in the metric case, however. Adapting certain ideas of Beck [5], Larcher managed to show that, again, we get arbitrarily close to the optimal bound in [47], thereby improving earlier results of Hofer and Larcher [35].

Theorem 3.8. *For almost all $\alpha \in [0,1)$ the discrepancy of the Halton–Kronecker sequence is upper bounded by*

$$D_N^*\left((\boldsymbol{z}_k(\infty))_{k\geq 0}\right) \ll_{\alpha,\delta} (\log N)^{2+\delta} \ll_\epsilon N^\epsilon.$$

In the other direction, we have $\ell = 1$ as a limiting case. This means, our perturbing sequence $\boldsymbol{c}$ consists of 1's entirely. As we will see in the next section, our technique of finding lower discrepancy bounds for $(\boldsymbol{z}_k(\ell))_{k\geq 0}$ leads to the study of certain subsequences of the pure Kronecker sequence $(\{m_k\alpha\})_{k\geq 0}$. Specifically, *evil Kronecker sequences* are linked to $(\boldsymbol{z}_k(1))_{k\geq 0}$, i.e. m_k refers to the sequence of *evil numbers* in this case. This, in turn, is the increasing sequence of non-negative integers whose sum of dyadic digits is even. Similarly, it turns out that the sequence $(m_k)_{k\geq 0}$ related to $(\boldsymbol{z}_k(\ell))_{k\geq 0}$ is the increasing sequence of non-negative integers with an even sum of digits in base 2^ℓ, i.e.

$$m_k = \mu_0 + \mu_1 2 + \mu_2 2^2 + \cdots, \qquad \mu_i \in \{0,1\}$$

with

$$\mu_0 + \mu_\ell + \mu_{2\ell} + \cdots \equiv 0 \bmod 2.$$

The evil Kronecker sequence was thoroughly studied by Aistleitner, Hofer, and Larcher in [2]. It needs to be mentioned that several techniques of the results presented below originate from this paper. From their results one can deduce the following bounds in the b.c.f.c. case.

Theorem 3.9. *For all irrational $\alpha \in [0,1)$ with b.c.f.c. we obtain*

$$D_N^*\left((\boldsymbol{z}_k(1))_{k\geq 0}\right) \ll N^{\log_4 3+\epsilon}, \qquad \epsilon > 0.$$

On the other hand, for $\alpha = 1/3 + \beta$ we have

$$D_N^*\left((\boldsymbol{z}_k(1))_{k\geq 0}\right) \gg N^{\log_4 3-\epsilon}, \qquad \epsilon > 0,$$

where $\beta = \sum_{l\geq 0} 4^{-2^l}$ denotes the special number introduced by Shallit [75].

Considering $\log_4 3 \approx 0.79\ldots$ evidently, Halton–Kronecker sequences yield much better discrepancy estimates than $(\boldsymbol{z}_k(1))_{k\geq 0}$, although the underlying concepts of those two limiting cases is very similar. This immediately raises the question of what happens in between, i.e. for $1 < \ell < \infty$.However, the answer we found to this question is not exactly what one would expect, see [37, Theorem 1.1].

Theorem 3.10. *Let $\ell \in \mathbb{N}$ and $\alpha \in (0,1)$ with b.c.f.c. Then the star discrepancy of the first N elements of the sequence $(\boldsymbol{z}(\ell)_k)_{k\geq 0}$ satisfies*

$$D_N^*\left((\boldsymbol{z}_k(\ell))_{k\geq 0}\right) \ll_\ell N^{a(\ell)+\epsilon},$$

for all $\epsilon > 0$, where

$$a(\ell) = \log_{2^\ell}\left(\cot\frac{\pi}{2(2^\ell+1)}\right). \tag{3.2}$$

In the other direction, we can proof the sharpness of this result in the b.c.f.c. case by utilizing Shallit's β again, cf. [37, Theorem 1.2].

Theorem 3.11. *Let $\ell \in \mathbb{N}$ and $\alpha = \frac{2^\ell}{2(2^\ell+1)} + \beta$. Then we have*

$$D_N^*\left((\boldsymbol{z}_k(\ell))_{k\geq 0}\right) \gg N^{a(\ell)-\epsilon}$$

for all $\epsilon > 0$, where $a(\ell)$ is given by (3.2).

Observe that $a(1) = \log_4 3$, hence this theorem is an extension of Theorem 3.9 indeed. However, considering that the discrepancy is much larger for $\ell = 1$ than for $\ell = \infty$ one would naturally expect $a(\ell)$ to decrease from $\log_4 3$ towards $1/2$, which is surprisingly not the case: $a(\ell)$ increases with respect to ℓ. This means that, if the densitiy of 1's in the first row of our generating matrix C decreases, the best possible bound for the star discrepancy of the hybrid sequence grows in the b.c.f.c. case. Figure 3.1 depicts a plot of $a(\ell)$ for $1 \leq \ell \leq 50$.

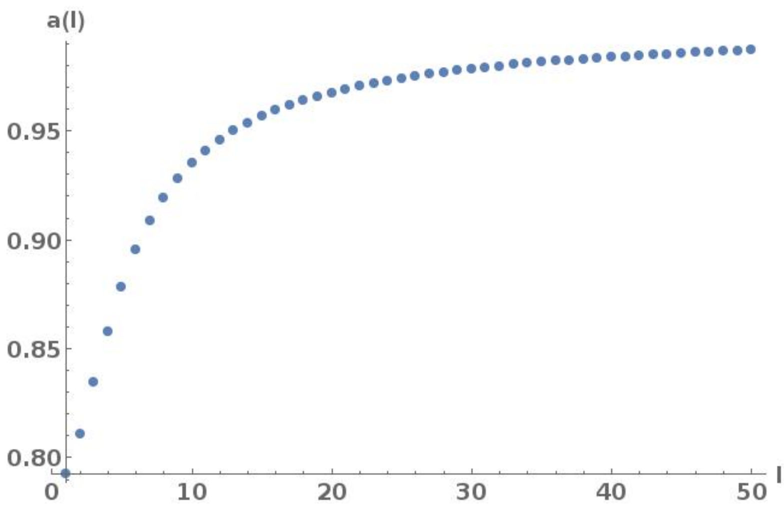

Figure 3.1: Plot of the exponent $a(\ell)$ for $1 \leq \ell \leq 50$.

It is not hard to check that

$$\lim_{\ell \to \infty} a(\ell) = 1.$$

Hence, the estimate from Theorem 3.10 approaches a trivial bound for huge ℓ. In this case, however, we can refer to Theorem 3.7, implying that the exponent of N experiences a sudden drop by approximately $1/2$ in the limit case $\ell = \infty$.

The paper [2] also considers the metric behavior of the evil Kronecker sequence. Again, from their result one may derive the theorem below.

Theorem 3.12. *For all $\epsilon > 0$ and almost all $\alpha \in (0,1)$ we have*

$$D_N^* \left((\boldsymbol{z}_k(1))_{k \geq 0} \right) \ll N^{1+\log_2 \Lambda + \epsilon}$$

as well as

$$D_N^* \left((\boldsymbol{z}_k(1))_{k \geq 0} \right) \gg N^{1+\log_2 \Lambda - \epsilon},$$

where $0.66130 < \Lambda < 0.66135$*. Hence, the exponents might be bounded by* 0.404 *and* 0.403*, respectively.*

Compared to the surprising behavior of the exponent $a(\ell)$, the situation seems to change completely from a metric point of view in the general setting $1 < \ell < \infty$. We will show the following result in Section 3.2.4, see [37, Theorem 1.4].

Theorem 3.13. *Let $\ell \in \mathbb{N}$. There exist real numbers $\Lambda_1(\ell)$ and $\Lambda_2(\ell)$ with*

$$0 \leq 1 + \log_{2^\ell} \Lambda_1(\ell) \leq 1 + \log_{2^\ell} \Lambda_2(\ell) \qquad \text{and} \qquad \lim_{\ell \to \infty} (1 + \log_{2^\ell} \Lambda_2(\ell)) = 0, \tag{3.3}$$

such that for almost all $\alpha \in (0,1)$ and all $\epsilon > 0$ we have

$$D_N^*((\boldsymbol{z}_k(\ell))_{k \geq 0}) \ll_{\ell,\alpha,\epsilon} N^{1+\log_{2^\ell} \Lambda_2(\ell)+\epsilon}, \tag{3.4}$$

and

$$D_N^*((\boldsymbol{z}_k(\ell))_{k \geq 0}) \gg N^{1+\log_{2^\ell} \Lambda_1(\ell)-\epsilon}, \tag{3.5}$$

Furthermore, upper and lower bounds of the exponents in the estimates from above and below, respectively, for small values of ℓ are given in Figure 3.2.

ℓ	1	2	3	4	5
$1 + \log_{2^\ell} \Lambda_1(\ell)$	0.40337	0.37489	0.34961	0.32651	0.30450
$1 + \log_{2^\ell} \Lambda_2(\ell)$	0.40348	0.37516	0.34962	0.32672	0.30599

Figure 3.2: Approximations of the exponents from Theorem 3.13.

Remark 3.14. Numerical experiments lead us to the conjecture that the exponents are decreasing in ℓ. Moreover, in the limit case $\ell = \infty$ we see that the behavior of $\Lambda_2(\ell)$ is in accordance to Theorem 3.8. I.e., in the case where the density of 1's in our perturbing sequence $\boldsymbol{c}$ is extremely sparse, (3.3) implicitly shows the optimality of the exponents.

As it has already been mentioned, the above theorems strongly rely on estimates of *lacunary trigonometric products* of the form

$$\Pi_{r,\boldsymbol{\gamma}}(\alpha) = \prod_{j=0}^{r-1} \left| \cos\left(2^j \alpha\pi + \gamma_j \frac{\pi}{2}\right) \right|, \tag{3.6}$$

where $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \gamma_2, \ldots) \in \{0,1\}^{\mathbb{N}}$, $\alpha \in (0,1)$ and $r \in \mathbb{N}$. Here, the term *lacunary* refers to the exponential growth of the argument of the cosine function. Since these are interesting subjects in their own right, we present them in the separate Section 3.2.5. More precisely, we will show (cf. [37, Theorem 3.1]).

Proposition 3.15. For all $r \in \mathbb{N}$ and all $l \in \mathbb{N}_0$ we have

$$\Pi_{r,\boldsymbol{c}^{(l)}}(\alpha) \ll_\ell 2^{-r} \left(\cot \frac{\pi}{2(2^\ell + 1)}\right)^{\frac{r}{\ell}}$$

for every $\alpha \in [0,1)$. Here, $\boldsymbol{c}^{(l)}$ denotes the perturbing sequence $\boldsymbol{c}$ shifted by l, i.e. $\boldsymbol{c}^{(l)} = (c_l, c_{l+1}, \ldots)$.

Moreover, this bound is asymptotically optimal in r, since for $l = 0$

$$\Pi_{\ell L,\boldsymbol{c}}\left(\frac{2^{\ell-1}}{2^\ell+1}\right) = 2^{-\ell L}\left(\cot\frac{\pi}{2(2^\ell+1)}\right)^L, \qquad L \in \mathbb{N}.$$

As a matter of fact, the quantities $\Lambda_1(\ell)$ and $\Lambda_2(\ell)$ occurring in Theorem 3.13 stem from the following metric result, which is originally found in [37, Proposition 1.6].

Proposition 3.16. Let $\ell \in \mathbb{N}$. We have

$$\int_0^1 \Pi_{\ell L,\boldsymbol{c}}(\alpha)\,\mathrm{d}\alpha \leq \big(\mu(\ell)\big)^L \tag{3.7}$$

for every $L \in \mathbb{N}$ with

$$\mu(\ell) = \frac{1}{4^\ell}\sum_{k=0}^{2^\ell-1}\left|\cos\left(\frac{(1+2k)\pi}{2^{\ell+1}}\right)\right|^{-1}.$$

Furthermore, there are positive real numbers $\Lambda_1(\ell)$ and $\Lambda_2(\ell)$ such that for every $\epsilon > 0$

$$\left(2^L\right)^{\log_2\Lambda_1(\ell)-\epsilon} \leq \int_0^1 \Pi_{\ell L,\boldsymbol{c}}(\alpha)\,\mathrm{d}\alpha \leq \left(2^L\right)^{\log_2\Lambda_2(\ell)+\epsilon} \tag{3.8}$$

for all $L > L_0(\ell,\epsilon)$.

In a more specified setting the above propositions have already occured in [2, 23, 24].

3.2.3 Sharp discrepancy bounds for α with b.c.f.c.

Here, we give the proofs for the upper and lower bounds for the star discrepancy of the perturbed Halton–Kronecker sequence in the case where α has b.c.f.c. Let us first focus on Theorem 3.10.

The upper bound – Theorem 3.10

We begin with one of the core estimates for the star discrepancy of $(\boldsymbol{z}_k(\ell))_{k\geq 0}$ which essentially separates the influence of the sequence $\boldsymbol{c}$ from diophantine properties of α via the product (3.6) and a term containing expressions of the form $\langle 2^l h\alpha\rangle$, respectively. Here,

$$\langle t\rangle = \min\{\{t\}, 1-\{t\}\}, \qquad t\in\mathbb{R}$$

denotes the distance of t to the nearest integer. Higher dimensional analogues over $\mathbb{Z}/p\mathbb{Z}$ with p prime in place of $\mathbb{Z}/2\mathbb{Z}$ of the proposition below are known to the author and are only more technical to derive. But as we do not want to divert the reader's attention from the core issues, we do not state this result in its full generality at this point and postpone it to Proposition 3.28.

Proposition 3.17 (Cf. [37, Proposition 2.1])**.** For all periods $\ell\in\mathbb{N}$ and every irrational $\alpha\in(0,1)$ the star discrepancy of $(\boldsymbol{z}_k(\ell))_{k\geq 0}$ satisfies

$$\begin{aligned} D_N^*((\boldsymbol{z}_k(\ell))_{k\geq 0}) \ll \frac{N}{K} + \frac{N}{H}\log N + \log^2 N + \\ + \sum_{l=1}^{\lfloor \log_2 K\rfloor}\sum_{h=1}^{\lfloor H/2^l\rfloor}\frac{1}{h}\left(\frac{1}{\langle 2^l h\alpha\rangle} + \sum_{r=0}^{\lfloor \log_2 N\rfloor - l} 2^r \Pi_{r,\boldsymbol{c}^{(l)}}(2^l h\alpha)\right), \end{aligned} \tag{3.9}$$

for all positive integers $H, K\leq N$, where $\Pi_{r,\boldsymbol{c}^{(l)}}$ is defined in (3.6).

Proof. First, we prove the inequality

$$\begin{aligned} D_N^*((\boldsymbol{z}_k(\ell))_{k\geq 0}) \ll \frac{N}{K} + \frac{N}{H}\log N + \log^2 N + \sum_{l=1}^{\lfloor \log_2 K\rfloor}\sum_{h=1}^{\lfloor H/2^l\rfloor}\frac{1}{h}\left(\frac{1}{\langle 2^l h\alpha\rangle}\right. \\ \left. + \left|\sum_{\substack{k=0\\ k=k_0+2k_1+\cdots}}^{\lfloor N/2^l\rfloor - 1}\exp\left(2\pi\mathrm{i}\left(2^l h\alpha k + \frac{k_0 c_l + k_1 c_{l+1} + \cdots}{2}\right)\right)\right|\right). \end{aligned} \tag{3.10}$$

To this end, we fix an arbitrary rectangle $J=[0,\theta)\times[0,\varphi)$ anchored at the origin. Furthermore, we consider the dyadic expansion of θ

$$\theta = 2^{-1}\theta_1 + 2^{-2}\theta_2 + \cdots$$

with $\theta_i\neq 1$ infinitely often. Subsequently, we choose $K\leq N$ and abbreviate $\mathsf{k}=\lfloor \log_2 K\rfloor$. On the basis of this we set $\Sigma_k(\theta)=\sum_{j=1}^{k}\theta_j 2^{-j}$ and define the intervals Θ and $J_\theta(l)$, $1\leq l\leq\mathsf{k}$, for $\beta_l=1$ by

$$J_\theta(l) := [\Sigma_{l-1}(\theta), \Sigma_l(\theta)), \qquad \Theta := \left[\Sigma_{\mathsf{k}}(\theta), \Sigma_{\mathsf{k}}(\theta)+2^{-\mathsf{k}}\right).$$

We extend the definition of the counting part $\mathcal{A}(\mathcal{S}, N, \boldsymbol{x})$ to arbitrary intervals in place of $[0,\boldsymbol{x})$ and, in order to faciliate notation, we abbreviate

$$\mathcal{A}(N, J_1\times J_2) := \mathcal{A}((\boldsymbol{z}_k(\ell))_{k\geq 0}, N, J_1\times J_2), \qquad J_1, J_2\subseteq[0,1).$$

It is immediately clear that

$$\bigcup_{\substack{l=1\\ \theta_l=1}} J_\theta(l) \subseteq [0,\theta) \subseteq \bigcup_{\substack{l=1\\ \theta_l=1}} J_\theta(l) \cup \Theta.$$

Using the additional fact that all the intervals $J_\theta(l)$ together with Θ are mutually disjoint, we easily obtain

$$\begin{aligned} |\mathcal{A}(N,J) - N\lambda_2(J)| \leq &\sum_{\substack{l=1,\\ \theta_l=1}}^{\mathsf{k}} |\mathcal{A}(N, J_\theta(l)\times[0,\varphi)) - N\lambda_2(J_\theta(l)\times[0,\varphi))| \\ &+ \max\left\{\mathcal{A}(N,\Theta\times[0,1)), N\lambda_2(\Theta\times[0,1))\right\}. \quad (3.11) \end{aligned}$$

Notice that Θ is an elementary interval in base 2 with volume $2^{-\mathsf{k}}$. As C is non-singular, $(\boldsymbol{z}_k(\ell))_{k\geq 0}$ is a $(0,1)$-sequence by Theorem 3.5 and, hence

$$\mathcal{A}(N,\Theta\times[0,1)) \leq \frac{N}{2^{\mathsf{k}}} + 1 = N\lambda_2(\Theta\times[0,1)) + 1.$$

Consequently,

$$\max\left\{\mathcal{A}(N,\Theta\times[0,1)), N\lambda_2(\Theta\times[0,1))\right\} \leq \frac{N}{2^{\mathsf{k}}} + 1 \ll \frac{N}{K}. \quad (3.12)$$

With a view to rewriting the first line of (3.11) the subsequent paragraphs emphasize on deriving certain conditions under which the elements of $(x_k(\ell))_{k\geq 0}$ lie in some $J_\theta(l)$. To this end we fix $l \leq \mathsf{k}$ such that $\theta_l = 1$. Furthermore, let $\sigma_0 + 2\sigma_1 + \cdots$ be the dyadic expansion of a non-negative integer σ. By the construction of our sequence it is easy to see that $x_{2\sigma+\varrho} \in J_\theta(l)$, $\varrho \in \{0,1\}$, if and only if

$$\begin{cases} s_{\boldsymbol{c}^{(1)}}(\sigma) = \sigma_0 c_1 + \sigma_1 c_2 + \cdots \equiv \theta_1 - \varrho \pmod 2, \\ \sigma_i = \theta_{i+2} \quad \text{for all} \quad 0 \leq i \leq l-3, \quad \text{and} \\ \sigma_{l-2} = 0. \end{cases}$$

Here, $s_{\boldsymbol{c}^{(j)}}(\cdot)$ denotes the weighted sum of digits in base 2 with weight sequence $\boldsymbol{c}$ shifted by $j \geq 1$, as indicated. The above set of conditions is equivalent to

$$\begin{cases} s_{\boldsymbol{c}^{(1)}}(\sigma) \equiv \theta_1 - \varrho \pmod 2, \\ \sigma \equiv R_{\theta,l} \pmod{2^{l-1}}, \end{cases}$$

with the integer $0 \leq R_{\theta,l} < 2^{l-1}$ given by

$$R_{\theta,l} = \theta_2 + 2\theta_3 + \cdots + 2^{l-2}\theta_l.$$

Since, obviously,

$$s_{\boldsymbol{c}^{(1)}}(\sigma) \equiv s_{\boldsymbol{c}^{(1)}}(R_{\theta,l}) + s_{\boldsymbol{c}^{(l)}}\left(\left\lfloor \frac{\sigma}{2^{l-1}} \right\rfloor\right) \pmod 2$$

the above conditions can be reformulated as

$$\begin{cases} \sigma \equiv R_{\theta,l} \pmod{2^{l-1}}, \\ s_{\boldsymbol{c}^{(l)}}\left(\left\lfloor \frac{\sigma}{2^{l-1}} \right\rfloor\right) \equiv \theta_1 - \varrho - s_{\boldsymbol{c}^{(1)}}(R_{\theta,l}) \pmod 2. \end{cases}$$

By a geometric sum argument it is evident that for any integer u we have $s_{\boldsymbol{c}^{(l)}}(u) \equiv v \bmod 2$ if and only if

$$\Sigma_{l,v}(u) := \frac{1}{2} \sum_{z\in\{0,1\}} \exp\left(2\pi\mathrm{i}\frac{z}{2}\left(s_{\boldsymbol{c}^{(l)}}(u) - v\right)\right) = 1.$$

In any other case $\Sigma_{l,v}(u) = 0$. Therefore, we may summarize the above paragraphs as

$$x_{2\sigma+\varrho} \in J_\theta(l) \iff \begin{cases} \sigma \equiv R_{\theta,l} \pmod{2^{l-1}}, \text{ and} \\ \Sigma_{l,\theta_1-\varrho-s_{\boldsymbol{c}^{(1)}}(R_{\theta,l})}\left(\left\lfloor \frac{\sigma}{2^{l-1}} \right\rfloor\right) = 1. \end{cases} \tag{3.13}$$

For fixed l and ϱ we introduce the increasing sequence $(\sigma_k^{(l,\varrho)})$ composed of all the integers solving (3.13). Since infinitely many elements of the sequence $\boldsymbol{c}$ are different from 0, this is an infinite sequence, indeed. Furthermore, we define the numbers $S^{(l,\varrho)}(N) = k_0 + 1$, where $2\sigma_{k_0}^{(l,\varrho)} + \varrho < N \le 2\sigma_{k_0+1}^{(l,\varrho)} + \varrho$. Since C is non-singular and $J_\theta(l)$ is an elementary interval in base 2 with volume 2^{-l} we have

$$\left\lfloor \frac{N}{2^l} \right\rfloor \le S^{(l,0)}(N) + S^{(l,1)}(N) \le \left\lfloor \frac{N}{2^l} \right\rfloor + 1. \tag{3.14}$$

Let us continue with (3.11). As a consequence of the above paragraph we immediately obtain

$$\begin{aligned} &|\mathcal{A}_N(J_\theta(l) \times [0,\varphi)) - N\lambda_2(J_\theta(l) \times [0,\varphi))| \\ &\le 1 + \sum_{\varrho\in\{0,1\}} \left| \#\left\{0 \le \nu < N : \nu \in \{\sigma_k^{(l,\varrho)} : k \ge 0\}, \{\nu\alpha\} \in [0,\varphi)\right\} - S^{(l,\varrho)}(N)\lambda_1([0,\varphi)) \right| \\ &\qquad \le 1 + D^*_{S^{(l,0)}(N)}\left((\{\sigma_k^{(l,0)}\alpha\})_{k\ge0}\right) + D^*_{S^{(l,1)}(N)}\left((\{\sigma_k^{(l,1)}\alpha\})_{k\ge0}\right). \end{aligned}$$

Considering this as well as (3.12) in (3.11) thus implies

$$|\mathcal{A}_N(J) - N\lambda_2(J)| \ll \frac{N}{K} + \log K + \sum_{\varrho\in\{0,1\}} \sum_{\substack{l=1 \\ \theta_l=1}}^{\mathsf{k}} D^*_{S^{(l,\varrho)}(N)}\left((\{\sigma_k^{(l,\varrho)}\alpha\})_{k\ge0}\right).$$

Subsequently, we invoke the Erdős–Turán inequality with $H \leq N$ and obtain

$$|\mathcal{A}_N(J)-N\lambda_2(J)| \ll \frac{N}{K}+\log K+\sum_{\varrho\in\{0,1\}}\sum_{\substack{l=1\\ \theta_l=1}}^{\mathsf{k}}\left(\frac{S^{(l,\varrho)}(N)}{\lfloor H/2^l\rfloor}+\sum_{h=1}^{\lfloor H/2^l\rfloor}\frac{1}{h}\left|\sum_{k=0}^{S^{(l,\varrho)}(N)-1}\mathrm{e}^{2\pi\mathrm{i}\sigma_k^{(l,\varrho)}h\alpha}\right|\right)$$

$$\ll \frac{N}{K}+\frac{N}{H}\log K+\sum_{\varrho\in\{0,1\}}\sum_{\substack{l=1\\ \theta_l=1}}^{\mathsf{k}}\sum_{h=1}^{\lfloor H/2^l\rfloor}\frac{1}{h}\left|\sum_{k=0}^{\lfloor N/2^l\rfloor-\delta_{\theta,l,\varrho}}\Sigma_{l,\theta_1-\varrho-s_{\boldsymbol{c}^{(1)}}(R_{\theta,l})}(k)\mathrm{e}^{2\pi\mathrm{i}(2^lk+2R_{\theta,l}+\varrho)h\alpha}\right|$$

$$= \frac{N}{K}+\frac{N}{H}\log K+\sum_{\varrho\in\{0,1\}}\sum_{\substack{l=1\\ \theta_\ell=1}}^{\mathsf{k}}\sum_{h=1}^{\lfloor H/2^l\rfloor}\frac{1}{h}\left|\sum_{k=0}^{\lfloor N/2^l\rfloor-\delta_{\theta,l,\varrho}}\Sigma_{l,\theta_1-\varrho-s_{\boldsymbol{c}^{(1)}}(R_{\theta,l})}(k)\mathrm{e}^{2\pi\mathrm{i}2^lh\alpha k}\right|.$$

for some $\delta_{\theta,l,\varrho} \in \{0,1\}$, where we used (3.13) and (3.14). It needs to be added that it is widely believed that an application of the Erdős–Turán inequality comes at a cost of $\log N$ (see, e.g. [7]). Since we are only interested in obtaining the right power of N regardless of the logarithmic terms, we may as well take advantage of the structural change towards exponential sums.

The innermost sum in the estimate above requires some further investigation. First of all, we intend to eliminate the dependence on ϱ. To this end, we insert the definition of $\Sigma_{l,\theta_1-\varrho-s_{\boldsymbol{c}^{(1)}}(R_{\theta,l})}(k)$, subsequently exchange the order of summation and apply the triangle inequality to get

$$\begin{aligned}
&\left|\sum_{k=0}^{\lfloor N/2^l\rfloor-\delta_{\theta,l,\varrho}}\Sigma_{l,\theta_1-\varrho-s_{\boldsymbol{c}^{(1)}}(R_{\theta,l})}(k)\mathrm{e}^{2\pi\mathrm{i}2^lh\alpha k}\right|\\
&\qquad\leq 1+\frac{1}{2}\left|\sum_{k=0}^{\lfloor N/2^l\rfloor-1}\mathrm{e}^{2\pi\mathrm{i}2^lh\alpha k}\sum_{z\in\{0,1\}}\mathrm{e}^{\pi\mathrm{i}z\left(s_{\boldsymbol{c}^{(l)}}(k)-\theta_1+\varrho+s_{\boldsymbol{c}^{(1)}}(R_{\theta,l})\right)}\right|\\
&\qquad\leq 1+\frac{1}{2}\sum_{z\in\{0,1\}}\left|\sum_{k=0}^{\lfloor N/2^l\rfloor-1}\mathrm{e}^{\pi\mathrm{i}z(-\theta_1+\varrho+s_{\boldsymbol{c}^{(1)}}(R_{\theta,l}))}\mathrm{e}^{2\pi\mathrm{i}\left(2^lkh\alpha+\frac{z}{2}s_{\boldsymbol{c}^{(l)}}(k)\right)}\right|\\
&\qquad= 1+\frac{1}{2}\sum_{z\in\{0,1\}}\left|\sum_{k=0}^{\lfloor N/2^l\rfloor-1}\mathrm{e}^{2\pi\mathrm{i}\left(2^lkh\alpha+\frac{z}{2}s_{\boldsymbol{c}^{(l)}}(k)\right)}\right|
\end{aligned}$$

For $z = 0$ the absolute value of the sum over k can be estimated by $(2\langle 2^lh\alpha\rangle)^{-1}$. Indeed, as α is irrational we immediately see

$$\left|\sum_{k=0}^{\lfloor N/2^l\rfloor-1}\mathrm{e}^{2\pi\mathrm{i}2^lkh\alpha}\right| \leq \frac{2}{|\mathrm{e}^{2\pi\mathrm{i}2^lh\alpha}-1|} = \frac{1}{\left|\frac{1}{2i}(\mathrm{e}^{\pi\mathrm{i}2^lh\alpha}-\mathrm{e}^{-\pi\mathrm{i}2^lh\alpha})\right|} = \frac{1}{|\sin(\pi 2^lh\alpha)|}$$

Following an observation from [45] we notice that $\sin(\pi 2^l h\alpha) = \sin(\pi\langle 2^l h\alpha\rangle)$. Furthermore, $\sin(\pi x) \geq x$ for all $0 \leq x \leq 1/2$ and, consequently,

$$\left|\sum_{k=0}^{\lfloor N/2^l\rfloor-1} \mathrm{e}^{2\pi\mathrm{i}2^l kh\alpha}\right| \leq \frac{1}{2\langle 2^l h\alpha\rangle}. \tag{3.15}$$

It thus remains to show

$$\left|\sum_{k=0}^{\lfloor N/2^l\rfloor-1} \mathrm{e}^{2\pi\mathrm{i}\left(2^l kh\alpha+\frac{1}{2}s_{\mathbf{c}^{(l)}}(k)\right)}\right| \leq \sum_{r=0}^{\lfloor\log_2 N\rfloor-l} 2^r \Pi_{r,\mathbf{c}^{(l)}}(2^l h\alpha).$$

but this easily follows by choosing $f(v) = 2^l vh\alpha + \frac{1}{2}s_{\mathbf{c}^{(l)}}(v)$ in Lemma 3.18 below. □

Lemma 3.18 (Cf. [37, Lemma 2.2])**.** *Let $f : \mathbb{N}_0 \to \mathbb{R}$ be a 2-additive function, i.e., for $v = v_0 + 2v_1 + 2^2 v_2 + \cdots$, $v_i \in \{0,1\}$, the function f satisfies*

$$f(v) = f(v_0) + f(2v_1) + f(2^2 v_2) + \cdots .$$

Then we have

$$\left|\sum_{v=0}^{V-1} \mathrm{e}^{2\pi\mathrm{i}f(v)}\right| \leq \sum_{r=0}^{\lfloor\log_2 V\rfloor} \prod_{j=0}^{r-1} \left|1 + \mathrm{e}^{2\pi\mathrm{i}f(2^j)}\right| = \sum_{r=0}^{\lfloor\log_2 V\rfloor} 2^r \prod_{j=0}^{r-1} \left|\cos\left(\pi f(2^j)\right)\right|$$

for all $V \in \mathbb{N}$.

Proof. We expand $V = V_0 + 2V_1 + \cdots + 2^{\lfloor\log_2 V\rfloor} V_{\lfloor\log_2 V\rfloor}$, $V_r \in \{0,1\}$, for all $0 \leq r \leq \lfloor\log_2 V\rfloor$. Since f is 2-additive we can estimate the sum on the left-hand side as follows

$$\begin{aligned}
\left|\sum_{v=0}^{V-1} \mathrm{e}^{2\pi\mathrm{i}f(v)}\right| &\leq \sum_{\substack{r=0\\V_r=1}}^{\lfloor\log_2 V\rfloor} \left|\sum_{k=0}^{2^r-1} \mathrm{e}^{2\pi\mathrm{i}\left(f(k)+\sum_{j=r}^{\lfloor\log_2 V\rfloor} f(2^j V_j)\right)}\right| \leq \sum_{r=0}^{\lfloor\log_2 V\rfloor} \left|\sum_{k=0}^{2^r-1} \mathrm{e}^{2\pi\mathrm{i}f(k)}\right| \\
&= \sum_{r=0}^{\lfloor\log_2 V\rfloor} \left|\sum_{v_0\in\{0,1\}} \cdots \sum_{v_{r-1}\in\{0,1\}} \mathrm{e}^{2\pi\mathrm{i}\left(f(v_0)+f(2v_1)+\cdots+f(2^{r-1}v_{r-1})\right)}\right| \\
&= \sum_{r=0}^{\lfloor\log_2 V\rfloor} \left|\prod_{j=0}^{r-1} \left(1 + \mathrm{e}^{2\pi\mathrm{i}f(2^j)}\right)\right|,
\end{aligned}$$

since, obviously, $f(0) = 0$. □

Considering Proposition 3.17 we require two more types of estimates: one focusing on the terms comprising $h\langle 2^l\alpha h\rangle$ relying on diophantine approximation properties of α, and one controlling the lacunary trigonometric products $\Pi_{r,\boldsymbol{c}^{(l)}}(\cdot)$. The latter is dealt with in Proposition 3.15. For the other part we invoke the lemma below.

Lemma 3.19. *Let H, K, N be positive integers with $H, K \leq N$ and let $\alpha \in \mathbb{R}$ have b.c.f.c. Then*

$$\sum_{l=1}^{\lfloor \log_2 K \rfloor} \sum_{h=1}^{\lfloor H/2^l \rfloor} \frac{1}{h\langle 2^l \alpha h\rangle} \ll_\alpha H \log K.$$

Moreover, for almost all $\alpha \in (0,1)$ in the sense of the Lebesgue measure we have

$$\sum_{l=1}^{\lfloor \log_2 K \rfloor} \sum_{h=1}^{\lfloor H/2^l \rfloor} \frac{1}{h\langle 2^l \alpha h\rangle} \ll_{\alpha,\epsilon} N^\epsilon$$

for all $\epsilon > 0$.

Proof. The first claim of this lemma can be found in [19, Proof of Theorem 2]. The second one is a consequence of [47, Lemma 3]. □

Proof of Theorem 3.10. We take Proposition 3.17 as our starting point. As a consequence of Proposition 3.15 and the first estimate from Lemma 3.19 we immediately obtain

$$\begin{aligned} D_N^*\left((\boldsymbol{z}_k(\ell))_{k\geq 0}\right) &\ll_{\ell,\alpha} \frac{N}{K} + \frac{N^{1+\epsilon}}{H} + N^\epsilon + H \log K \\ &\qquad + \left(\cot \frac{\pi}{2(2^\ell+1)}\right)^{\frac{\log_2 N}{\ell}} \log H \log K \\ &\ll \frac{N}{K} + \frac{N^{1+\epsilon}}{H} + N^\epsilon + H\log K + N^{a(\ell)+\epsilon}. \end{aligned}$$

The terms involving H or K can be balanced out by choosing $H = \lfloor\sqrt{N}\rfloor$ and $K = N$, for instance, and the result follows considering $a(\ell) \geq \log_4 3 \geq 1/2$. □

Remark 3.20. The result of Theorem 3.10 is valid for an even wider scope of α's. As a matter of fact, we can show the following: If α is of finite type $\varsigma \geq 1$ then

$$D_N^*\left((\boldsymbol{z}_k(\ell))_{k\geq 0}\right) \ll_{\ell,\alpha,\epsilon} N^{1-\frac{1}{\varsigma+1}+\epsilon} + N^{a(\ell)+\epsilon} \tag{3.16}$$

for all $\epsilon > 0$. Here, we say α is of finite type ς iff for all $\nu > 0$ there exists a constant $c_{\alpha,\nu}$ such that

$$h^{\varsigma+\nu}\langle h\alpha\rangle \geq c_{\alpha,\nu}$$

for all integers $h \neq 0$. Balancing both terms of (3.16) yields a bound on ς depending on the period ℓ. We shall give a proof of this result within the paragraphs below. Before we do so, it is worth mentioning that almost all α are of finite type 1. Therefore, Theorem 3.10 even holds for almost all α as well. As a metric result, however, this bound is far from being optimal considering Theorem 3.13.

We begin the proof of (3.16) by deriving an analogous version of Lemma 3.19. As a first step towards this direction we cite the lemma below, which can be found in [38].

Lemma 3.21 (Gap lemma). *Let $I \in \mathbb{N}$ and let $x_1, x_2, \ldots, x_I$ be real numbers. Furthermore, let f be a non-negative and non-increasing function on $[0,1]$. Under the assumption that there exists a parameter $0 < \delta \leq 1/2$ such that*

$$\langle x_i\rangle \geq \delta \qquad \text{and} \qquad \langle x_i - x_j\rangle \geq \delta$$

for all $i, j \in [\![I]\!]$, $i \neq j$, we have that for every $i \in [\![I]\!]$ there are at most two indices $j \in [\![I]\!]$ with $|\langle x_i\rangle - \langle x_j\rangle| < \delta$. Moreover,

$$\sum_{i=1}^{I} f(\langle x_i\rangle) \leq 2 \sum_{j=1}^{\lfloor 1/(2\delta)\rfloor} f(j\delta).$$

The proof of the lemma below is due to Roswitha Hofer via personal communication.

Lemma 3.22. *Let H, K, N be positive integers with $H, K \leq N$ and let α be of finite type ς. Then*

$$\sum_{l=1}^{\lfloor \log_2 K\rfloor} \sum_{h=1}^{\lfloor H/2^l\rfloor} \frac{1}{h\langle 2^l\alpha h\rangle} \ll H^{\varsigma-1+\epsilon}K$$

for all $\epsilon > 0$.

Proof. Let $\epsilon > 0$. With the choice $\nu = \epsilon/2$ we have by definition

$$\langle 2^l\alpha h\rangle \geq c_{\alpha,\epsilon}(2^l h)^{-\varsigma-\frac{\epsilon}{2}}.$$

For all $1 \leq l \leq \lfloor \log_2 K \rfloor$ we define a function f_l by $f_l(t) = 1/(t(t+1))$ for $1 \leq t < \lfloor H/2^l \rfloor$ and $f_l(\lfloor H/2^l \rfloor) = 1/\lfloor H/2^l \rfloor$. As in the proof of [59, Lemma 3] we may rewrite

$$\sum_{l=1}^{\lfloor \log_2 K \rfloor} \sum_{h=1}^{\lfloor H/2^l \rfloor} \frac{1}{h\langle 2^l \alpha h \rangle} = \sum_{l=1}^{\lfloor \log_2 K \rfloor} \sum_{t=1}^{\lfloor H/2^l \rfloor} f_l(t) \sum_{h=1}^{t} \frac{1}{\langle 2^l \alpha h \rangle}.$$

Subsequently, we invoke the gap lemma with

$$\delta = c_{\alpha,\epsilon} \left(2^l t\right)^{-\varsigma - \frac{\epsilon}{2}}.$$

Notice that this choice fulfills the necessary prerequisites. Therefore,

$$\sum_{h=1}^{t} \frac{1}{\langle 2^l \alpha h \rangle} \ll_{\alpha,\epsilon} \left(2^l t\right)^{\varsigma + \frac{\epsilon}{2}} \sum_{h=1}^{\lfloor 1/(2\delta) \rfloor} \frac{1}{h} \ll \left(2^l t\right)^{\varsigma+\epsilon}.$$

Considering additionally

$$\sum_{t=1}^{\lfloor H/2^l \rfloor} f_l(t) t^{\varsigma+\epsilon} \ll_\epsilon \lfloor H/2^l \rfloor^{\varsigma - 1 + \epsilon}$$

we finally arrive at

$$\begin{aligned} \sum_{l=1}^{\lfloor \log_2 K \rfloor} \sum_{h=1}^{\lfloor H/2^l \rfloor} \ll_{\alpha,\epsilon} \sum_{l=1}^{\lfloor \log_2 K \rfloor} 2^{l(\varsigma+\epsilon)} \sum_{t=1}^{\lfloor H/2^l \rfloor} f_l(t) t^{\varsigma+\epsilon} \\ \ll \sum_{l=1}^{\lfloor \log_2 K \rfloor} 2^{l(\varsigma+\epsilon)} \left(\frac{H}{2^l}\right)^{\varsigma-1+\epsilon} \ll H^{\varsigma-1+\epsilon} K. \end{aligned}$$

□

The inequality (3.16) follows from Proposition 3.17 together with Lemma 3.22. Indeed, in complete analogy to the proof of Theorem 3.10 we have

$$D_N^* \left((\boldsymbol{z}_k(\ell))_{k \geq 0} \right) \ll_\ell \frac{N}{K} + \frac{N}{H}^{1+\epsilon} + N^\epsilon + H^{\varsigma-1+\epsilon} K + N^{a(\ell)+\varepsilon}.$$

By choosing $H = K = N^{\frac{1}{\varsigma+1}}$ the sought bound is obtained.

The lower bound – Theorem 3.11

First of all we consider certain subsequences $(\{m_k\alpha\})_{k\geq 0}$ of the pure one-dimensional Kronecker sequence. Here, the sequence $(m_k)_{k\geq 0} = (m_k(\ell))_{k\geq 0}$ denotes the increasing sequence of non-negative integers with an even sum of digits in base 2^ℓ, which have already been introduced in the paragraph following Theorem 3.8. For $\ell = 1$ these numbers are better known as *evil numbers* and the discrepancy of the associated *evil Kronecker sequence* has been thoroughly studied in [2] and yields the exponents $\log_4(3) \pm \epsilon$, which coincides with our values $a(1) \pm \epsilon$. The relation between $(\{m_k\alpha\})_{k\geq 0}$ and $(\boldsymbol{z}_k(\ell))_{k\geq 0}$ will be highlighted in the proof of Theorem 3.11.

The proposition below provides a lower bound for the discrepancy of $(\{m_k\alpha\})_{k\geq 0}$ and its proof builds upon and extends several techniques from [2].

Proposition 3.23 (Cf. [37, Proposition 2.4])**.** Let $M = 2^{\ell L-1}$, $L \in \mathbb{N}$. The star discrepancy of the first M elements of the sequence $(\{m_k\alpha\})_{k\geq 0}$ is lower-bounded by

$$D_M^*(\{m_k\alpha\}) \geq 2^{\ell L-3}\Pi_{\ell L,\boldsymbol{c}}(\alpha) - \frac{|\sin(2^{\ell L}\pi\alpha)|}{4\sin(\pi\alpha)}.$$

Proof. As a first step we apply the Koksma–Hlawka inequality (Theorem 1.6) to the function $f(x) = \mathrm{e}^{2\pi\mathrm{i}x}$ with the integration nodes $\mathcal{P} = \{\{m_k\alpha\} : 0 \leq k < M\}$. Observe that

$$MR_{M,\mathcal{P}}(f) = M\,|I(f) - Q_{M,\mathcal{P}}(f)| = \left|\sum_{k=0}^{M-1} \mathrm{e}^{2\pi\mathrm{i}m_k\alpha}\right|.$$

Moreover, it is known that f has variation $c_f = 4$. Consequently,

$$D_M^*(\{m_k\alpha\}) \geq \frac{1}{4}\left|\sum_{k=0}^{M-1} \mathrm{e}^{2\pi\mathrm{i}m_k\alpha}\right|.$$

Subsequently, we focus on rewriting the exponential sum on the right-hand side. To this end, we notice that for $\mu_j \in \{0,1\}$, $0 \leq j < \ell L$, we have

$$\frac{1}{2}\sum_{z\in\{0,1\}} \mathrm{e}^{2\pi\mathrm{i}\frac{z}{2}\sum_{j=0}^{\ell L-1}\mu_j c_j} = \begin{cases} 1, & \text{if } \mu_0 + \mu_\ell + \cdots + \mu_{(L-1)\ell} \equiv 0 \pmod 2 \\ 0, & \text{else.} \end{cases}$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{M-1} \mathrm{e}^{2\pi\mathrm{i}m_k\alpha} &= \sum_{\substack{m=0\\ m=\mu_0+2\mu_1+\cdots}}^{2^{\ell L}-1} \mathrm{e}^{2\pi\mathrm{i}m\alpha}\frac{1}{2}\sum_{z\in\{0,1\}} \mathrm{e}^{2\pi\mathrm{i}\frac{z}{2}\sum_{j=0}^{\ell L-1}\mu_j c_j}\\ &= \frac{1}{2}\sum_{\substack{m=0\\ m=\mu_0+2\mu_1+\cdots}}^{2^{\ell L}-1} \mathrm{e}^{2\pi\mathrm{i}m\alpha}\mathrm{e}^{2\pi\mathrm{i}\frac{1}{2}\sum_{j=0}^{\ell L-1}\mu_j c_j} + \frac{1}{2}\sum_{m=0}^{2^{\ell L}-1}\mathrm{e}^{2\pi\mathrm{i}m\alpha}\\ &= P_1 + P_0.\end{aligned}$$

In order to rewrite the absolute value of P_1 as the claimed trigonometric product we proceed as follows:

$$\begin{aligned}2P_1 &= \sum_{\substack{m=0\\ m=\mu_0+2\mu_1+\cdots}}^{2^{\ell L}-1} \mathrm{e}^{2\pi\mathrm{i}m\alpha}\mathrm{e}^{2\pi\mathrm{i}\frac{1}{2}\left(\mu_0+\mu_\ell+\cdots+\mu_{(L-1)\ell}\right)}\\ &= \sum_{\substack{m=0\\ m=\mu_0+2\mu_1+\cdots}}^{2^{\ell L}-1} \mathrm{e}^{2\pi\mathrm{i}m\alpha}\mathrm{e}^{2\pi\mathrm{i}\frac{1}{2}\sum_{\nu=0}^{L-1}\left(\mu_{\nu\ell}+2\mu_{\nu\ell+1}+\cdots+2^{\ell-1}\mu_{(\nu+1)\ell-1}\right)}\\ &= \sum_{\substack{m=0\\ m=\tilde\mu_0+2^l\tilde\mu_1+\cdots}}^{2^{\ell L}-1} \mathrm{e}^{2\pi\mathrm{i}\alpha\left(\tilde\mu_0+2^\ell\tilde\mu_1+\cdots+2^{(L-1)\ell}\tilde\mu_{L-1}\right)}\mathrm{e}^{2\pi\mathrm{i}\frac{1}{2}(\tilde\mu_0+\tilde\mu_1+\cdots+\tilde\mu_{L-1})}\\ &= \prod_{\nu=0}^{L-1}\sum_{\tilde\mu=0}^{2^\ell-1}\mathrm{e}^{2\pi\mathrm{i}\tilde\mu\left(2^{\ell\nu}\alpha+\frac{1}{2}\right)}.\end{aligned}$$

For fixed $0 \le \nu < L$ we can simplify the absolute value of each of the above factors, giving

$$\begin{aligned}
\left|\sum_{\tilde\mu=0}^{2^\ell-1} \mathrm{e}^{2\pi\mathrm{i}\tilde\mu\left(2^{\ell\nu}\alpha+\frac{1}{2}\right)}\right| &= \left|\sum_{\tilde\mu=0}^{2^\ell-1}(-1)^{\tilde\mu}\mathrm{e}^{2\pi\mathrm{i}2^{\ell\nu}\alpha\tilde\mu}\right| = \left|\frac{1-\mathrm{e}^{2\pi\mathrm{i}2^{\ell\nu}\alpha 2^\ell}}{1+\mathrm{e}^{2\pi\mathrm{i}2^{\ell\nu}\alpha}}\right| \\
&= \frac{1}{|2\cos(2^{\ell\nu}\alpha\pi)|}\left|1-\mathrm{e}^{2\pi\mathrm{i}2^{\ell\nu}\alpha}\right|\left|\sum_{\tilde\mu=0}^{2^\ell-1}\mathrm{e}^{2\pi\mathrm{i}2^{\ell\nu}\alpha\tilde\mu}\right| \\
&= \left|\tan(2^{\ell\nu}\alpha\pi)\right|\left|\sum_{\mu=-2^{\ell-1}+1}^{2^{\ell-1}}\mathrm{e}^{2\pi\mathrm{i}2^{\ell\nu}\alpha(\mu+2^{\ell-1}-1)}\right| \\
&= \left|\tan(2^{\ell\nu}\alpha\pi)\right|\left|\sum_{\mu=-2^{\ell-1}+1}^{2^{\ell-1}}\mathrm{e}^{\pi\mathrm{i}2^{\ell\nu}\alpha(2\mu-1)}\right| \\
&= \left|\tan(2^{\ell\nu}\alpha\pi)\right|\left|\sum_{\mathrm{u}\subseteq\{0,1,\dots,\ell-1\}}\mathrm{e}^{\pi\mathrm{i}2^{\ell\nu}\alpha\left(\sum_{j\in\mathrm{u}}2^j-\sum_{j\in\{0,1,\dots,l-1\}\setminus\mathrm{u}}2^j\right)}\right| \\
&= \left|\tan(2^{\ell\nu}\alpha\pi)\right|\left|\sum_{\mathrm{u}\subseteq\{0,1,\dots,\ell-1\}}\prod_{j\in\mathrm{u}}\mathrm{e}^{\pi\mathrm{i}2^{\ell\nu+j}\alpha}\prod_{j\in\{0,1,\dots,l-1\}\setminus\mathrm{u}}\mathrm{e}^{-\pi\mathrm{i}2^{\ell\nu+j}\alpha}\right| \\
&= \left|\tan(2^{\ell\nu}\alpha\pi)\right|\prod_{\mu=0}^{\ell-1}\left|\mathrm{e}^{\pi\mathrm{i}2^{\ell\nu+\mu}\alpha}+\mathrm{e}^{-\pi\mathrm{i}2^{\ell\nu+\mu}\alpha}\right| \\
&= \left|2\sin(2^{\ell\nu}\alpha\pi)\right|\prod_{\mu=1}^{\ell-1}|2\cos(2^{\ell\nu+\mu}\alpha\pi)|.
\end{aligned}$$

Similarly to the above lines we can rewrite $|P_0|$ as

$$2|P_0| = \prod_{\nu=0}^{L}\left|\sum_{\tilde\mu=0}^{2^\ell-1}\mathrm{e}^{2\pi\mathrm{i}\tilde\mu 2^{\ell\nu}\alpha}\right| = \prod_{\nu=0}^{L}\left|\sum_{\mu=-2^{\ell-1}+1}^{2^{\ell-1}}\mathrm{e}^{2\pi\mathrm{i}2^{\ell\nu}(2\mu-1)}\right|$$

to find that

$$2|P_0| = \prod_{k=0}^{\ell L-1}\left|2\cos\left(2^k\alpha\pi\alpha\right)\right| = \frac{|\sin(2^{\ell L}\pi\alpha)|}{\sin(\pi\alpha)}$$

The identities for $|P_0|$ and $|P_1|$ together with the triangle inequality yields the estimate from the claim. □

Remark 3.24. The minuend in the claim of Proposition 3.23 can be replaced by $(16\langle\alpha\rangle)^{-1}$. Indeed, observe that $2|P_0|$ is of the same form as the left-hand side of (3.15) with $l = 0$ and can therefore be treated analogously.

Proof of Theorem 3.11. Let $N = 2^{\ell L}$, $L \in \mathbb{N}$. First of all, we find a lower bound for the star discrepancy by specifying the interval under consideration for the first coordinate. I.e.

$$\begin{aligned} D_N^*((\boldsymbol{z}_k(\ell))_{k\geq 0}) &= \sup_{0\leq\theta,\varphi\leq 1} \left|\mathcal{A}\left((\boldsymbol{z}_k(\ell))_{k\geq 0}, N, [0,\theta)\times[0,\varphi)\right) - N\lambda_2\left([0,\theta)\times[0,\varphi)\right)\right| \\ &\geq \sup_{0\leq\varphi\leq 1} \left|\mathcal{A}\left((\boldsymbol{z}_k(\ell))_{k\geq 0}, N, [0,1/2)\times[0,\varphi)\right) - \frac{N}{2}\lambda_1\left([0,\varphi)\right)\right|. \end{aligned}$$

Furthermore, observe that $x_m(\ell) \in [0, 1/2)$ if and only if m is an element of the sequence $(m_k)_{k\geq 0}$. Therefore, the above inequality implies

$$\begin{aligned} D_N^*((\boldsymbol{z}_k(\ell))_{k\geq 0}) &\gg D^*_{2^{\ell L-1}}((\{m_k\alpha\})_{k\geq 0}) \\ &\gg 2^{\ell L-1}\Pi_{\ell L,\boldsymbol{c}^{(\ell)}}(\alpha) - \frac{|\sin(2^{\ell L}\pi\alpha)|}{2\sin(\pi\alpha)} \\ &\geq 2^{\ell L-1}\Pi_{\ell L,\boldsymbol{c}^{(\ell)}}(\alpha) - \frac{1}{4\langle\alpha\rangle}. \end{aligned} \tag{3.17}$$

where we used Proposition 3.23 in the second and Remark 3.24 in the last step. For α and β as stated in the claim we define $\alpha_0 = \alpha - \beta$ as well as $\delta_l = \{2^l\beta\}$. Furthermore, commonly known trigonometric identities imply

$$\begin{aligned} |\sin(2^{\ell\nu}\alpha\pi)| &= |\sin(2^{\ell\nu}\alpha_0\pi)\cos(\delta_{\ell\nu}\pi) \pm \cos(2^{\ell\nu}\alpha_0\pi)\sin(\delta_{\ell\nu}\pi)|, \\ |\cos(2^{\ell\nu+\mu}\alpha\pi)| &= |\cos(2^{\ell\nu+\mu}\alpha_0\pi)\cos(\delta_{\ell\nu+\mu}\pi) \pm \sin(2^{\ell\nu+\mu}\alpha_0\pi)\sin(\delta_{\ell\nu+\mu}\pi)|. \end{aligned}$$

Considering this as well as

$$|\sin(2^{\ell\nu}\alpha_0\pi)| = \left|\cos\left(\frac{\pi}{2^{\ell+1}+2}\right)\right|, \quad \text{and } |\cos(2^{\ell\nu}\alpha_0\pi)| = \left|\sin\left(\frac{\pi}{2^{\ell+1}+2}\right)\right|$$

we further obtain

$$\begin{aligned}
2^{\ell L}\Pi_{\ell L,\boldsymbol{c}}(\alpha) &= N^{a(\ell)}\Pi_{\ell L,\boldsymbol{c}}(\alpha)\left(\Pi_{\ell L,\boldsymbol{c}}(\alpha_0)\right)^{-1} \\
&= N^{a(\ell)}\prod_{\nu=0}^{L-1}\left|\frac{\sin\left(2^{\ell\nu}\alpha\pi\right)}{\sin\left(2^{\ell\nu}\alpha_0\pi\right)}\right|\prod_{\mu=1}^{\ell-1}\left|\frac{\cos\left(2^{\ell\nu+\mu}\alpha\pi\right)}{\cos\left(2^{\ell\nu+\mu}\alpha_0\pi\right)}\right| \\
&= N^{a(\ell)}\prod_{\nu=0}^{L-1}\left|\cos(\delta_{\ell\nu}\pi)\pm\sin(\delta_{\ell\nu}\pi)\tan\left(\frac{\pi}{2(2^n+1)}\right)\right| \\
&\quad\times\prod_{\mu=1}^{\ell-1}\left|\cos(\delta_{\ell\nu+\mu}\pi)\pm\sin(\delta_{\ell\nu+\mu}\pi)\tan\left(\frac{2^{\mu}(2^{\ell\nu}+1)\pi}{2(2^n+1)}-\frac{2^{\mu}\pi}{2(2^n+1)}\right)\right| \\
&= N^{a(\ell)}\prod_{\nu=0}^{L-1}\left|\cos(\delta_{\ell\nu}\pi)\pm\sin(\delta_{\ell\nu}\pi)\tan\left(\frac{\pi}{2(2^n+1)}\right)\right| \\
&\quad\times\prod_{\mu=1}^{\ell-1}\left|\cos(\delta_{\ell\nu+\mu}\pi)\pm\sin(\delta_{\ell\nu+\mu}\pi)\tan\left(\frac{2^{\mu}\pi}{2(2^n+1)}\right)\right| \\
&=: N^{a(\ell)}\prod_{\nu=0}^{L-1}\left(S_\nu\prod_{\mu=1}^{\ell-1}C_{\nu,\mu}\right), \qquad (3.18)
\end{aligned}$$

where we made use of the fact that $(2^{\ell\nu}+1)/(2^n+1)$ can be expressed as a geometric sum with integer valued summands as well as of the periodicity of the tangent function.

Since $1-\cos(x\pi)\le\sqrt{6}x$ and $\sin x\le x$ for all $x\ge 0$ we further obtain

$$S_\nu \ge 1-\delta_{\ell\nu}\left(\sqrt{6}+\pi\tan\left(\frac{\pi}{2(2^\ell+1)}\right)\right) =: 1-\delta_{\ell\nu}\gamma_0(\ell)$$

by the triangle inequality. In the same spirit we can derive

$$C_{\nu,\mu}\ge 1-\delta_{\ell\nu+\mu}\left(\sqrt{6}+\pi\tan\left(\frac{2^\mu\pi}{2(2^\ell+1)}\right)\right) =: 1-\delta_{\ell\nu+\mu}\gamma_\mu(\ell),\quad 1\le\mu<\ell.$$

On the other hand we can define the numbers $T_0,T_1,\ldots,T_L$ for each fixed ℓ by the relations

$$T_0=\inf_{\nu\ge 0}\left|\frac{\sin\left(2^{\ell\nu}\alpha\pi\right)}{\sin\left(2^{\ell\nu}\alpha_0\pi\right)}\right|,\qquad\text{and}\qquad T_\mu=\inf_{\nu\ge 0}\left|\frac{\cos\left(2^{\ell\nu}\alpha\pi\right)}{\cos\left(2^{\ell\nu}\alpha_0\pi\right)}\right|,\quad 1\le\mu<n.$$

These numbers are bounded by positive constants from below, as

$$\inf\left\{|\{2^l\alpha\}-\kappa| : \kappa\in\{0,1,1/2\}, l\in\mathbb{N}_0\right\}>0$$

due to the special structure of β.

We may thus continue with (3.18) by finding a constant $\overline{c}(\ell) > 0$ such that $\max\{1 - \gamma_\mu(\ell)x, T_\mu\} \geq \mathrm{e}^{-\overline{\gamma}(\ell)x}$ for all $x \geq 0$ and every $\mu \in \{0, 1, \ldots, \ell - 1\}$. Hence,

$$2^{\ell L}\Pi_{\ell L,\boldsymbol{c}}(\alpha) \gg N^{a(\ell)} \prod_{\nu=0}^{L-1} \prod_{\mu=0}^{\ell-1} \max\left\{(1 - \delta_{\ell\nu+\mu}\gamma_\mu(\ell)), T_\mu\right\}$$

$$\geq N^{a(\ell)} \prod_{k=0}^{\ell L-1} \mathrm{e}^{-\overline{\gamma}(\ell)\delta_k} \geq N^{a(\ell)}\mathrm{e}^{-\underline{\gamma}(\ell)\log \ell L} \gg N^{a(\ell)-\epsilon}, \qquad \text{with } \underline{\gamma}(\ell) > 0,$$

where we used $\sum_{k=0}^{K} \delta_k \leq \tilde{\gamma} \log K$ for an absolute constant $\tilde{\gamma} > 0$ and K large enough. Together with (3.17) the above estimate concludes the proof. □

3.2.4 Tight metric discrepancy bounds

The main objective of this section is to rigorously proof Theorem 3.13. This entails a variety of subtasks, i.e., we need to establish the upper and lower metric discrepancy bounds (3.4) and (3.5) and verify the limit statement in (3.3), and briefly describe the procedure leading to the explicit bounds given in Figure 3.2. For the first of these tasks we heavily rely on the ideas and strategies developed in [2], which were refined and extended in [3]. The last of these tasks is postponed to the proof of Proposition 3.16 in Section 3.2.5 and discussed in Section 3.2.6.

In their recent paper [3] Aistleitner and Larcher focus on metric discrepancy bounds for sequences of the form $(\{a_k\alpha\})_{k\geq1}$ with a_k growing at most polynomially in k. Naturally, this perfectly fits into our setting and we shall make use of their result below (see [2, Theorem 3]) for establishing the subsequent Lemma 3.26, which in turn is essential for the proof of Theorem 3.13.

Lemma 3.25. *Let $(a_k)_{k\geq1}$ be a sequence of integers such that for some $t \in \mathbb{N}$ we have $a_k \leq k^t$ for all k large enough. Assume there exists a number $\tau \in (0, 1)$ and a strictly increasing sequence $(B_L)_{L\geq1}$ of positive integers with $(B')^L \leq B_L \leq B^L$ for some reals B', B with $1 < B' < B$, such that for all $\epsilon > 0$ and all $L > L_0(\epsilon)$ we have*

$$\int_0^1 \left| \sum_{k=1}^{B_L} \mathrm{e}^{2\pi \mathrm{i} a_k \alpha} \right| \mathrm{d}\alpha > B_L^{\tau-\epsilon}.$$

Then for almost all $\alpha \in [0, 1)$ and all $\epsilon > 0$ for the star discrepancy of the sequence $(\{a_k\alpha\})_{k\geq1}$ we have

$$D_N^* \left((\{a_k\alpha\})_{k\geq1}\right) \gg N^{\tau-\epsilon}.$$

Lemma 3.26 (Cf. [37, Lemma 2.7])**.** *Let $\ell \in \mathbb{N}$. If there exists a number $\tau = \tau(\ell)$ such that the inequality*

$$\int_{[0,1]} \left(2^{\ell L} \Pi_{\ell L, \boldsymbol{c}}(\alpha) - \frac{|\sin(2^{\ell L}\pi\alpha)|}{\sin(\pi\alpha)} \right) \mathrm{d}\alpha \geq 2^{\ell L(\tau - \epsilon)}$$

holds for every $\epsilon > 0$ and for L large enough, then

$$D_N^*((\boldsymbol{z}_k(\ell)_{k\geq 0})) \gg N^{\tau-\varepsilon}.$$

Proof. This immediately follows from Lemma 3.25 and the inequality in (3.17) together with the proof of 3.23. □

Proof of Theorem 3.13. Let us verify the lower bound first. With a view to Lemma 3.26 we put $N = 2^{\ell L}$, $L \in \mathbb{N}$ and seek estimates for the L_1-norm of $\Pi_{\ell,\boldsymbol{c}}(\cdot)$ and $\sin(2^{\ell L}\pi\cdot)/\sin(\pi\cdot)$ from below and above, respectively. The first one is dealt with in Proposition 3.16. For the second one we can use

$$\int_{[0,1)} \frac{|\sin(2^k \pi x)|}{\sin(\pi x)} \mathrm{d}x \ll k, \qquad k \geq 1.$$

Indeed, we may estimate

$$\int_{[0,1)} \frac{|\sin(2^k\pi x)|}{\sin(\pi x)} \mathrm{d}x = \sum_{l=0}^{2^k-1} \int_{\frac{l}{2^k}}^{\frac{l+1}{2^k}} \frac{|\sin(2^k\pi x)|}{\sin(\pi x)} \mathrm{d}x \leq \frac{1}{2^k} \sum_{l=0}^{2^k-1} \frac{1}{\sin(l2^{-k}\pi)}.$$

Subsequently, we use the symmetry of the sine function as well as the trivial estimate $\sin(\pi x) \geq x$ valid for all $0 \leq x \leq 1/2$ to verify the assertion

$$\frac{1}{2^k} \sum_{l=0}^{2^k-1} \frac{1}{\sin(l2^{-k}\pi)} \ll \frac{1}{2^{k-1}} \sum_{l=0}^{2^{k-1}} \frac{1}{\sin(l2^{-k}\pi)} \ll \sum_{l=0}^{2^{k-1}} \frac{1}{l} \ll k.$$

The proof of the upper bound is considerably more delicate, since we do not have a comparably strong tool as Lemma 3.25 at our disposal. However, certain strategies from [2] work in our favour. As a first step, we invoke Proposition 3.17 with $K = H = N$ to find that

$$D_N^*\left((\boldsymbol{z}_k(\ell))_{k\geq 0}\right) \ll N^\epsilon + \sum_{l=1}^{\lfloor \log_2 N \rfloor} \sum_{h=1}^{\lfloor N/2^l \rfloor} \frac{1}{h} \left(\frac{1}{\langle 2^l h\alpha \rangle} + \sum_{r=0}^{\lfloor \log_2 N \rfloor - l} 2^r \Pi_{r,\boldsymbol{c}^{(l)}}(2^l h\alpha) \right).$$

Considering Lemma 3.19 in the above inequality it remains to show

$$\sum_{l=1}^{\lfloor \log_2 N \rfloor} \sum_{h=1}^{\lfloor N/2^l \rfloor} \frac{1}{h} \sum_{r=0}^{\lfloor \log_2 N \rfloor - l} 2^r \Pi_{r,\boldsymbol{c}^{(l)}}(2^l h\alpha) \ll_{\alpha,\epsilon,\ell} N^{\log_{2^\ell}(\Lambda_2(\ell))+1+\epsilon}$$

for all $\epsilon > 0$ and almost all $\alpha \in (0,1)$ in the sense of the Lebesgue measure.

To this end, we first of all get rid of the superscript (l) in $\boldsymbol{c}^{(l)}$. Let us set $\kappa(l) \equiv \ell - l \pmod{n}$ and assume $r \geq \kappa(l)$. Since $\boldsymbol{c}$ has period ℓ we immediately see that

$$\begin{aligned}\Pi_{r,\boldsymbol{c}^{(l)}}(2^l h\alpha) &= \prod_{j=0}^{r-1} \left|\cos\left(2^{j+l} h\alpha\pi + c_{j+l}\frac{\pi}{2}\right)\right| \\ &= \prod_{j=-\kappa(l)}^{r-\kappa(l)-1} \left|\cos\left(2^{j+l+\kappa(l)} h\alpha\pi + c_j\frac{\pi}{2}\right)\right| \leq \Pi_{r-\kappa l,\boldsymbol{c}}(2^{l+\kappa(l)} h\alpha).\end{aligned}$$

Consequently,

$$\begin{aligned}\sum_{r=0}^{\lfloor \log_2 N\rfloor - l} 2^r \Pi_{r,\boldsymbol{c}^{(l}}(2^l h\alpha) &\ll 2^{\kappa(l)} + \sum_{r=\kappa(l)}^{\lfloor \log_2 N\rfloor - l} 2^{r-\kappa(l)} 2^{\kappa(l)} \Pi_{r-\kappa(l),\boldsymbol{c}}(2^{l+\kappa(l)} h\alpha) \\ &\ll_\ell 1 + \sum_{r=0}^{\lfloor \log_2 N\rfloor - l-\kappa(l)} 2^r 2^{\kappa(l)} \Pi_{r,\boldsymbol{c}}(2^{l+\kappa(l)} h\alpha) \\ &\ll_\ell 1 + \sum_{j=0}^{\lfloor (\log_2 N)/\ell\rfloor} 2^{\ell j} \sum_{k=0}^{\ell-1} 2^{k+\kappa(l)} \Pi_{\ell j,\boldsymbol{c}}(2^{l+\kappa(l)} h\alpha).\end{aligned}$$

These considerations, in turn, lead to the estimate

$$\begin{aligned}&\sum_{l=1}^{\lfloor \log_2 N\rfloor} \sum_{h=1}^{\lfloor N/2^l\rfloor} \frac{1}{h} \sum_{r=0}^{\lfloor \log_2 N\rfloor - l} 2^r \Pi_{r,\boldsymbol{c}^{(l)}}(2^l h\alpha) \\ &\ll_\ell (\log N)^2 + \sum_{j=0}^{\lfloor (\log_2 N)/\ell\rfloor} 2^{\ell j} \sum_{l=1}^{\lfloor \log_2 N\rfloor} \sum_{h=1}^{N} \frac{1}{h} \sum_{k=0}^{\ell-1} 2^{k+\kappa(l)} \Pi_{\ell j,\boldsymbol{c}}(2^{l+\kappa(l)} h\alpha). \quad (3.19)\end{aligned}$$

We fix $\epsilon > 0$ and set $\mu_\ell := \lceil (1 + \log_{2^\ell}(\Lambda_2(\ell)))^{-1}\rceil$. Proposition 3.16 implies

$$\int_{[0,1)} \left(\sum_{l=1}^{\ell j\mu_\ell} \sum_{h=1}^{2^{\ell j\mu_\ell}} \frac{1}{h} \sum_{k=0}^{\ell-1} 2^{k+\kappa(l)} \Pi_{\ell j,\boldsymbol{c}}(2^{l+\kappa(l)} h\alpha) \right) \mathrm{d}\alpha \leq c(\ell) \left(2^{\ell j}\right)^{\log_{2^\ell}(\Lambda_2(\ell)) + \frac{\epsilon}{2}} \quad (3.20)$$

for all $j > j_0(\ell,\epsilon)$, where $c(\ell) > 0$ is an absolute constant only depending on ℓ. For all positive integers j and for $\epsilon > 0$ we define the events

$$G_j := \left\{ \alpha \in (0,1) : \sum_{l=1}^{\ell j\mu_\ell} \sum_{h=1}^{2^{\ell j\mu_\ell}} \frac{1}{h} \sum_{k=0}^{\ell-1} 2^{k+\kappa(l)} \Pi_{\ell j,\boldsymbol{c}}(2^{l+\kappa(l)} h\alpha) > c(\ell) \left(2^{\ell j}\right)^{\log_{2^\ell}(\Lambda_2(\ell)) + \epsilon} \right\}.$$

By Markov's inequality they occur with probability

$$\mathbb{P}(G_j) \le (2^{\ell j})^{-\frac{\epsilon}{2}}, \qquad \text{for all } \; j > j_0(\ell, \epsilon).$$

Hence, the probabilities $\mathbb{P}(G_j)$ are summable with respect to j. As a consequence of the Borel–Cantelli lemma we obtain that for almost all α the events G_j occur only j-finitely often. In other words, we have

$$\sum_{l=1}^{\ell j \mu_\ell} \sum_{h=1}^{2^{\ell j \mu_\ell}} \frac{1}{h} \sum_{k=0}^{\ell-1} 2^{k+\kappa(l)} \Pi_{\ell j, \boldsymbol{c}}(2^{l+\kappa(l)} h\alpha) \le c(\ell) \left(2^{\ell j}\right)^{\log_{2^\ell}(\Lambda_2(\ell))+\epsilon}, \qquad j \ge j_1(\ell, \epsilon).$$

We now return to the entire expression under investigation, i.e. (3.19). Let $\epsilon > 0$, $N > 2^{\ell \mu_\ell j_1(\ell,\epsilon)}$, and $\alpha \in (0,1)$ such that the above inequality holds. We split the entire sum over j in (3.19) at $M = \lceil \log_{2^\ell} N/\mu_\ell \rceil \ge j_1(\ell, \epsilon)$. Subsequently, we invoke the first part of Proposition 3.16 to the first M summands, resulting in

$$\sum_{j=0}^{M-1} 2^{\ell j} \sum_{l=1}^{\lfloor \log_2 N \rfloor} \sum_{h=1}^{N} \frac{1}{h} \sum_{k=0}^{\ell-1} 2^{k+\kappa(l)} \Pi_{\ell j, \boldsymbol{c}}(2^{l+\kappa(l)} h\alpha) \ll_\ell 2^{\ell M} N^\epsilon \ll_\ell N^{1+\log_{2^\ell}(\Lambda_2(\ell))+\epsilon}.$$

The proof of the metric upper bound may then be concluded by estimating

$$\begin{aligned} &\sum_{j=M}^{\lfloor (\log_2 N)/\ell \rfloor} 2^{\ell j} \sum_{l=1}^{\lfloor \log_2 N \rfloor} \sum_{h=1}^{N} \frac{1}{h} \sum_{k=0}^{\ell-1} 2^{k+\kappa(l)} \Pi_{\ell j, \boldsymbol{c}}(2^{l+\kappa(l)} h\alpha) \\ &\qquad \le \sum_{j=M}^{\lfloor (\log_2 N)/\ell \rfloor} 2^{\ell j} \sum_{l=1}^{\ell j \mu_\ell} \sum_{h=1}^{2^{\ell j \mu_\ell}} \frac{1}{h} \sum_{k=0}^{\ell-1} 2^{k+\kappa(l)} \Pi_{\ell j, \boldsymbol{c}}(2^{l+\kappa(l)} h\alpha) \\ &\qquad \ll_\ell \sum_{j=M}^{\lfloor (\log_2 N)/\ell \rfloor} (2^{\ell j})^{1+\log_{2^\ell}(\Lambda_2(\ell))+\epsilon} \ll_\ell N^{1+\log_{2^\ell}(\Lambda_2(\ell))+\epsilon}. \end{aligned}$$

It remains to verify the limit statement in (3.3). From the proof of Proposition 3.16 it is evident that

$$\Lambda_2(\ell) \le \max_{x \in [0,1]} \Phi_{\ell,1}(x) = \mu(\ell) = \frac{1}{4^\ell} \sum_{k=0}^{2^\ell - 1} \left| \cos\left(\frac{(1+2k)\pi}{2^{\ell+1}} \right) \right|^{-1}.$$

Therefore, it suffices to show

$$\lim_{\ell \to \infty} \log_{2^\ell} \mu(\ell) = -1.$$

To this end we rewrite

$$\log_{2^\ell}\mu(\ell) = -1 + \log_{2^\ell}\left(\frac{1}{2^\ell}\sum_{k=0}^{2^\ell-1}\frac{1}{|\cos(\pi(1/2+k)/2^\ell)|}\right)$$

$$= -1 + \log_{2^\ell}\left(\frac{1}{2^{\ell-1}}\sum_{k=0}^{2^{\ell-1}-1}\frac{1}{\cos(\pi(1/2+k)/2^\ell)}\right)$$

$$= -1 + \frac{1}{\log 2}\log\left(\left(\frac{1}{2^{\ell-1}}\sum_{k=0}^{2^{\ell-1}-1}\frac{1}{\cos(\pi(1/2+k)/2^\ell)}\right)^{\frac{1}{\ell}}\right).$$

Obviously,

$$\left(\frac{1}{2^{\ell-1}}\sum_{k=0}^{2^{\ell-1}-1}\frac{1}{\cos(\pi(1/2+k)/2^\ell)}\right)^{\frac{1}{\ell}} \geq 1$$

and, hence, $\mu(\ell) \geq -1$.

In the other direction we can make use of the trivial estimate $\sin(\pi x/2) \geq x$ for $x \in [0,1]$ to obtain further

$$\sum_{k=0}^{2^{\ell-1}-1}\frac{1}{\cos(\pi(1/2+k)/2^\ell)} = \sum_{k=0}^{2^{\ell-1}-1}\frac{1}{\sin(\pi(1/2+k)/2^\ell)}$$

$$\leq \sum_{k=0}^{2^{\ell-1}-1}\frac{1}{(1/2+k)/2^{\ell-1}} = 2^{\ell-1}\sum_{k=0}^{2^{\ell-1}-1}\frac{1}{1/2+k} \leq 2^{\ell-1}(2+\ell\log 2).$$

Considering this in the original expression we thus obtain

$$\left(\frac{1}{2^{\ell-1}}\sum_{k=0}^{2^{\ell-1}-1}\frac{1}{\cos(\pi(1/2+k)/2^\ell)}\right)^{\frac{1}{\ell}} \leq \ell^{\frac{1}{\ell}}2^{\frac{1}{\ell}}(\log 2)^{\frac{1}{\ell}} \stackrel{\ell\to\infty}{\longrightarrow} 1.$$

□

3.2.5 Lacunary trigonometric products

We aim for the upper bound of $\Pi_{r,\boldsymbol{c}}(\alpha)$ as given in Proposition 3.15 as well as for a *bad* example for α to verify the sharpness of this estimate. As we have already seen in Section 3.2.3, this contributes largely to the proofs of

Theorems 3.10 and 3.11 and shall thus conclude the study of discrepancy bounds for α with b.c.f.c.

Subsequently, we focus on the metric results for the aforementioned trigonometric products and give a proof for Proposition 3.16, thus providing the last essential ingredient to the study of metric discrepancy bounds in Section 3.2.4.

Estimates in the general case

Let us approach the proof of Proposition 3.15 first. The case $\ell = 1$, i.e. $\boldsymbol{c} = (111\ldots)$, has already appeared in [24]. In this case two viable strategies are known to treat such products: one by Fouvry and Mauduit [24] and another one is due to Gel′fond [25]. For our purposes, i.e. $\boldsymbol{c}$ being of the particular form (3.1), numerical experiences convinced us to pursue the latter in [37].

To this end, we require some notation and initial remarks. We define a system of functions $\{f_\nu :\ \nu \geq 0\}$ with $f_\nu : [0,1] \to [0,1]$, where

$$f_0(x) = x, \qquad f_1(x) = 2x\sqrt{1-x^2}, \qquad f_\nu = f_1 \circ f_{\nu-1}(x),\ \nu \geq 2.$$

Furthermore, we abbreviate $g(x) = \sqrt{1-x^2}$. The role of these functions is revealed by taking $x = |\sin y|$. Observe that g now corresponds to a transition to $|\cos y|$ and f_1 corresponds to doubling the angle y.

In what follows we consider the function

$$G = G_\ell = f_0 \cdot \prod_{\nu=1}^{\ell-1} g \circ f_\nu = \frac{f_\ell}{2^\ell\sqrt{1-f_0^2}}.$$

Notice that at

$$\xi = \xi(\ell) = \sin\left(\frac{2^\ell \pi}{2(2^\ell+1)}\right) \tag{3.21}$$

the function G evaluates to

$$G(\xi) = \frac{1}{2^\ell}\cot\left(\frac{\pi}{2(2^\ell+1)}\right).$$

Moreover, $\xi = f_\ell(\xi)$. It is an evident observation that G and ξ are closely related to the trigonometric product and the *bad* example for α from Proposition 3.15, respectively.

Originally, Gel′fond considered products of iterates of a function with certain properties. The basic inherent idea was to show that if one indivdual factor grows too big then the next iterate, i.e. the consecutive factor, is small enough to make up for it. With significantly more technical effort we apply this strategy to the function G. The lemma below generalizes Gel′fond's approach and immediately implies Proposition 3.15.

Lemma 3.27 (Cf. [37, Lemma 3.2])**.** *Let $\ell \in \mathbb{N}$ and ξ be given as in (3.21). For all $x \in [0,1]$ either*

$$G(x) \le G(\xi) \quad \text{or} \quad G(x)\,(G \circ f_\ell)\,(x) \le \big(G(\xi)\big)^2.$$

Proof. Notice that the result for $\ell = 1$ has been obtained by Gel′fond in [25, Lemme II]. Therefore, we confine ourselves to $\ell > 1$. In what follows we verify the first inequality for $x \le \xi$ and, subsequently, the second inequality for $x > \xi$. This is done by distinguishing between several cases w.r.t. x and by using basic tools from fundamental analysis.

On $[0,1]$ we define the function

$$\tilde{G}_\ell(y) := G\left(\left|\sin\left(\frac{y\pi}{2}\right)\right|\right) = \frac{|\sin(\frac{2^\ell y\pi}{2})|}{2^\ell \cos(\frac{y\pi}{2})}$$

and immediately observe $\tilde{G}_\ell(2^\ell/(2^\ell+1)) = G(\sin(2^\ell\pi/(2(2^\ell+1))))$. Let us now begin with the case distinction.

Case 1: $y \in [0, (2^\ell - 1)/2^\ell]$.

In order to prove $\tilde{G}_\ell(y) \le G(\xi)$ we use the trivial estimate

$$\frac{|\sin(\frac{2^\ell y\pi}{2})|}{2^\ell \cos(\frac{y\pi}{2})} \le \frac{1}{2^\ell \cos(\frac{(2^\ell-1)\pi}{2^{\ell+1}})}$$

and subsequently show

$$\cos\left(\frac{(2^\ell-1)\pi}{2^{\ell+1}}\right) \ge \cot\left(\frac{2^\ell\pi}{2(2^\ell+1)}\right)$$

or, equivalently,

$$\sin\left(\frac{\pi}{2^{\ell+1}}\right) \ge \tan\left(\frac{\pi}{2(2^\ell+1)}\right).$$

To this end we set $z = 2^{-\ell}$ and observe that $z \in [0, 1/4]$. We may now rewrite the sought inequality as

$$h_1(z) := \sin\left(\frac{z\pi}{2}\right) \ge \tan\left(\frac{z\pi}{2(z+1)}\right) =: h_2(z).$$

For $z=0$ we have equality and for $z\in[0,1/4]$ we observe that $h_1(z),h_2(z)\geq 0$. Moreover, $h_1'(z)\geq h_2'(z)$, i.e.

$$1\leq \cos\left(\frac{z\pi}{2}\right)(z+1)\cdot\cos^2\left(\frac{z\pi}{2(z+1)}\right)(z+1).$$

Indeed, in what follows we show that each of the two factors above (separated by the dot) is greater or equal to 1.

Let us begin by verifying $\cos(z\pi/2)(z+1)\geq 1$. Equality holds for $z=0$ and the derivative of the left-hand side satisfies

$$-\sin\left(\frac{z\pi}{2}\right)\frac{(z+1)\pi}{2}+\cos\left(\frac{z\pi}{2}\right)\geq\cos\left(\frac{\pi}{8}\right)-\sin\left(\frac{\pi}{8}\right)\frac{5\pi}{8}>0$$

whenever $z\in[0,1/4]$.

Similarly, we obtain $\cos^2\left(\frac{z\pi}{2(z+1)}\right)(z+1)\geq 1$ for the second factor, since equality holds for $z=0$ and the derivative of the left hand side, i.e.

$$-2\cos\left(\frac{z\pi}{2(z+1)}\right)\sin\left(\frac{\pi}{2(z+1)}\right)\frac{\pi}{2(z+1)}+\cos^2\left(\frac{z\pi}{2(z+1)}\right),$$

is positive for $z\in[0,1/4]$. This can be derived in the same spirit as above after splitting up $[0,1/4]$ at $z=1/5$.

Case 2: $y\in[(2^\ell-1)/2^\ell,2^\ell/(2^\ell+1)]$.

In this case we parametrize y by $y=\frac{2^\ell}{2^\ell+1}-\frac{z}{2^\ell(2^\ell+1)}$ with $z\in[0,1]$ and obtain

$$\left|\sin\left(\frac{2^\ell y\pi}{2}\right)\right|=\sin\left(\frac{2^n\pi}{2(2^n+1)}+\frac{z\pi}{2(2^n+1)}\right).$$

In the subsequent paragraph we thus aim for the inequality

$$\frac{\sin\left(\frac{2^\ell\pi}{2(2^\ell+1)}+\frac{z\pi}{2(2^\ell+1)}\right)}{\cos\left(\frac{2^\ell\pi}{2(2^\ell+1)}-\frac{z\pi}{2^{\ell+1}(2^\ell+1)}\right)}\leq\tan\left(\frac{2^\ell\pi}{2(2^\ell+1)}\right).$$

It is an easy observation that equality holds for $z=0$. Furthermore, we can show that the derivative of the left-hand side is negative for $z\in[0,1]$. This will immediately follow once we have established the inequality

$$\cos\left(\frac{(2^\ell+z)\pi}{2^{\ell+1}(2^\ell+1)}\right)\sin\left(\frac{(2^\ell+z)\pi}{2(2^\ell+1)}\right)$$
$$\geq 2^\ell\cos\left(\frac{(2^\ell+z)\pi}{2(2^\ell+1)}\right)\sin\left(\frac{(2^\ell+z)\pi}{2^{\ell+1}(2^\ell+1)}\right)\quad(3.22)$$

for all $z \in [0,1]$ as $\cos\left(\frac{2^\ell \pi}{2(2^\ell+1)} - \frac{z\pi}{2^{\ell+1}(2^\ell+1)}\right) = \sin\left(\frac{(2^\ell+z)\pi}{2^{\ell+1}(2^\ell+1)}\right)$. We begin by demonstrating that the above inequality (3.22) is satisfied for $z=0$. Notice that

$$\cos^2\left(\frac{\pi}{2(2^\ell+1)}\right) \geq 2^\ell \sin^2\left(\frac{\pi}{2(2^\ell+1)}\right) \quad \Leftrightarrow \quad 1 \geq (2^\ell+1)\sin^2\left(\frac{\pi}{2(2^\ell+1)}\right).$$

This in turn is the case iff

$$\frac{\eta}{\eta+1} \geq \sin^2\left(\frac{\eta\pi}{2(\eta+1)}\right),$$

where $\eta = 2^{-\ell}$ and, thus, $\eta \in (0, 1/4]$. The last inequality holds as we have equality for $\eta = 0$ and the derivative w.r.t. η of the left-hand side is greater than the one of the right-hand side, since $2/\pi \geq \sin(\pi/5) \geq \sin(\eta\pi/(\eta+1))$.

To finally verify (3.22) for all $z \in [0,1]$ we compute the derivatives of both sides and observe that the one of the left-hand side oughtweighs the other one, since, obviously,

$$\sin\left(\frac{(2^\ell+z)\pi}{2^{\ell+1}(2^\ell+1)}\right)\sin\left(\frac{(2^\ell+z)\pi}{2(2^\ell+1)}\right)(4^\ell-1) \geq 0.$$

Case 3: $y \in [2^\ell/(2^\ell+1), 1]$.
In this case we need to show the more complicated estimate $\tilde{G}_\ell(y)\tilde{G}_\ell(2^\ell y) \leq (\tilde{G}_\ell(2^\ell/(2^\ell+1))^2$. The interval under consideration can be parametrized by $z \mapsto 2^\ell/(2^\ell+1) + z/(4^\ell(2^\ell+1))$, $z \in [0, 4^\ell]$, and we may rewrite

$$\left|\sin\left(\frac{2^\ell y\pi}{2}\right)\right| = \sin\left(\frac{(2^\ell - z/2^\ell)\pi}{2(2^\ell+1)}\right), \qquad \left|\cos\left(\frac{2^\ell y\pi}{2}\right)\right| = \cos\left(\frac{(2^\ell - z/2^\ell)\pi}{2(2^\ell+1)}\right),$$
$$\cos\left(\frac{y\pi}{2}\right) = \cos\left(\frac{(2^\ell + z/4^\ell)\pi}{2(2^\ell+1)}\right).$$

In order to be able to handle $|\sin(4^n y\pi/2)|$ we require one further case distinction.
Case 3a: $z \in [0,1]$.
Here, $|\sin(4^\ell y\pi/2)| = \sin((2^\ell+z)\pi/(2(2^\ell+1)))$. We need to derive the following inequality

$$h_3(z)h_4(z) \leq \tan^2\left(\frac{2^\ell\pi}{2(2^\ell+1)}\right), \tag{3.23}$$

where $h_3(z) = \frac{\sin((2^\ell+z)\pi/(2(2^\ell+1)))}{\cos((2^\ell-z/2^\ell)\pi/(2(2^\ell+1)))}$ and $h_4(z) = \frac{\sin((2^\ell-z/2^\ell)\pi/(2(2^\ell+1)))}{\cos((2^\ell+z/4^\ell)\pi/(2(2^\ell+1)))}$. Obviously, $h_3(z), 0 \leq h_4(z) \geq 0$ and for $z=0$ we even have equality in (3.23). In the following we show that the derivative of the left-hand side is negative for all $z \in [0,1]$. As a matter of fact, this is a consequence of

$$\frac{(h_3(z)h_4(z))'}{h_3(z)h_4(z)} = \frac{h_3'(z)}{h_3(z)} + \frac{h_4'(z)}{h_4(z)} \le 0,$$

which in turn can be rewritten as

$$\begin{aligned}\frac{2\cdot 4^\ell(2^\ell+1)}{\pi}&\frac{(h_3(z)h_4(z))'}{h_3(z)h_4(z)}\\ &= 4^\ell \cot\left(\frac{(2^\ell+z)\pi}{2(2^\ell+1)}\right) - 2^\ell \tan\left(\frac{(2^\ell - z/2^\ell)\pi}{2(2^\ell+1)}\right)\\ &\quad - 2^\ell \cot\left(\frac{(2^\ell - z/2^\ell)\pi}{2(2^\ell+1)}\right) + \tan\left(\frac{(2^\ell + z/4^\ell)\pi}{2(2^\ell+1)}\right) \le 0.\end{aligned}$$

Here we made use of the identities

$$\begin{aligned}h_3'(z) &= \frac{\pi}{2^{\ell+1}(2^\ell+1)}\\ &\quad\times \frac{2^\ell \cos\left(\frac{(2^\ell-z/2^\ell)\pi}{2(2^\ell+1)}\right)\cos\left(\frac{(2^\ell+z)\pi}{2(2^\ell+1)}\right) - \sin\left(\frac{(2^\ell-z/2^\ell)\pi}{2(2^\ell+1)}\right)\sin\left(\frac{(2^\ell+z)\pi}{2(2^\ell+1)}\right)}{\cos^2\left(\frac{(2^\ell-z/2^\ell)\pi}{2(2^\ell+1)}\right)},\\ h_4'(z) &= -h_3'(-z2^{-\ell})2^{-\ell}.\end{aligned}$$

For $z=0$ we have $\frac{(h_3(z)h_4(z))'}{h_3(z)h_4(z)} \le 0$ due to the proof of (3.22). For arbitrary $z \in (0,1)$ we have

$$\begin{aligned}2^\ell \cot\left(\frac{(2^\ell+z)\pi}{2(2^\ell+1)}\right)&\left(2^\ell - \frac{\cot\left(\frac{(2^\ell-z/2^\ell)\pi}{2(2^\ell+1)}\right)}{\cot\left(\frac{(2^\ell+z)\pi}{2(2^\ell+1)}\right)}\right)\\ &\le \tan\left(\frac{(2^\ell-z/2^\ell)\pi}{2(2^\ell+1)}\right)\left(2^\ell - \frac{\tan\left(\frac{(2^\ell+z/4^\ell)\pi}{2(2^\ell+1)}\right)}{\tan\left(\frac{(2^\ell-z/2^\ell)\pi}{2(2^\ell+1)}\right)}\right).\end{aligned}$$

Indeed, as a consequence of (3.22) we obtain

$$0 \le 2^\ell \cot\left(\frac{(2^\ell+z)\pi}{2(2^\ell+1)}\right) \le \tan\left(\frac{(2^\ell-z/2^\ell)\pi}{2(2^\ell+1)}\right).$$

Furthermore, we have

$$2^\ell - \frac{\cot\left(\frac{(2^\ell-z/2^\ell)\pi}{2(2^\ell+1)}\right)}{\cot\left(\frac{(2^\ell+z)\pi}{2(2^\ell+1)}\right)} \le 2^\ell - \frac{\tan\left(\frac{(2^\ell+z/4^\ell)\pi}{2(2^\ell+1)}\right)}{\tan\left(\frac{(2^\ell-z/2^\ell)\pi}{2(2^\ell+1)}\right)}$$

since its equivalent version

$$\tan\left(\frac{(2^\ell+z)\pi}{2(2^\ell+1)}\right) \geq \tan\left(\frac{(2^\ell+z/4^\ell)\pi}{2(2^\ell+1)}\right)$$

is obviously satisfied.

It remains to show

$$2^\ell - \frac{\tan\left(\frac{(2^\ell+z/4^\ell)\pi}{2(2^\ell+1)}\right)}{\tan\left(\frac{(2^\ell-z/2^\ell)\pi}{2(2^\ell+1)}\right)} \geq 2^\ell - \frac{\tan\left(\frac{(2^\ell+1/4^\ell)\pi}{2(2^\ell+1)}\right)}{\tan\left(\frac{(2^\ell-1/2^\ell)\pi}{2(2^\ell+1)}\right)} \geq 0.$$

The first inequality is evident and for the second one we consider the inequality below, which is obtained by setting $\eta = 2^{-\ell}$. I.e.,

$$\frac{(1-\eta)}{\eta}\cos\left(\frac{\eta\pi}{2}\right)\sin\left(\frac{\eta(1-\eta)\pi}{2}\right) \geq \sin\left(\frac{\eta^2\pi}{2}\right).$$

This inequality is satisfied for $\eta = 0$ as well as for $\eta = 1/4$. The right-hand side is monotonically increasing on $[0, 1/4]$, while the left-hand side is decreasing, as both $(1-\eta)^2\cos(\eta\pi/2)$ as well as $\sin(\eta(1-\eta)\pi/2)/(\eta(1-\eta))$ are decreasing and non-negative.

Case 3b: $z \in [1, 4^\ell]$.

We exploit the trivial fact $|\sin(4^\ell y\pi/2)| \leq 1$ and, hence, it remains to show that

$$\tan\left(\frac{(2^\ell-z/2^\ell)\pi}{2(2^\ell+1)}\right)\frac{1}{\cos\left(\frac{(2^\ell+z/4^\ell)\pi}{2(2^\ell+1)}\right)} \leq \tan^2\left(\frac{2^\ell\pi}{2(2^\ell+1)}\right).$$

For $z = 1$ the inequality follows from the previous case. Moreover, for $z \to 4^\ell$ the left-hand side tends to 2^ℓ. Since $2^\ell\cos^2\left(\frac{2^\ell\pi}{2(2^\ell+1)}\right) \leq \sin^2\left(\frac{2^\ell\pi}{2(2^\ell+1)}\right)$ (see (3.22)) the sought inequality is satisfied for $z = 4^\ell$ too. Once again, we need to check whether the left-hand side is decreasing or, equivalently, if

$$2^{\ell+1}\cot\left(\frac{(2^\ell+z/4^\ell)\pi}{2(2^\ell+1)}\right) \geq \sin\left(\frac{(2^\ell-z/2^\ell)\pi}{2^\ell+1}\right), \qquad z \in (1, 4^\ell).$$

This, however, is true since we have equality at the right end point $z = 4^\ell$ and since the derivative of the left-hand side is dominated by the one of the right-hand side, as clearly

$$-\frac{1}{\sin^2\left(\frac{(2^\ell+z/4^\ell)\pi}{2(2^\ell+1)}\right)} \leq -\cos\left(\frac{(2^\ell-z/2^\ell)\pi}{2^\ell+1}\right).$$

□

Metric estimates

Let us now tackle the proof of Proposition 3.16. The basic framework has already been provided by Fouvry and Mauduit, and by Aistleitner, Hofer and Larcher, and has been generalized by Hofer and the author in [37]. The entire proof consists of three main parts. First of all, we determine a recursively defined function $\Phi_{\ell,j}$, $j \leq L$, subject to the integral equation

$$\int_0^1 \Pi_{\ell L,\boldsymbol{c}}(\alpha)\,\mathrm{d}\alpha = \int_0^1 \Phi_{\ell,j}(\alpha)\Pi_{\ell(L-j),\boldsymbol{c}}(\alpha)\,\mathrm{d}\alpha, \tag{3.24}$$

cf. [24]. Secondly, we make use of this recurrence and invoke techniques from [23] and [2] to find an upper bound for $\Phi_{\ell,1}$, thus proving (3.7). Finally, we follow the basic approach from [2] to guarantee the existence of $\Lambda_1(\ell)$ and $\Lambda_2(\ell)$, thereby explaining how the approximative values of the exponents $1+\log_{2^\ell}\Lambda_i(\ell)$, $i=1,2$, in Figure 3.2 can be obtained.

Let us begin by deriving the recurrence (3.24). We do so by demonstrating the first step, i.e. for $j=1$, and the general version follows from iteratively applying the arguments below. We may rewrite the left-hand side as follows

$$\begin{aligned}
\int_0^1 \Pi_{\ell L,\boldsymbol{c}}(\alpha)\,\mathrm{d}\alpha &= \int_0^1 \Pi_{\ell,\boldsymbol{c}}(\alpha)\Pi_{\ell(L-1),\boldsymbol{c}}(2^\ell\alpha)\,\mathrm{d}\alpha \\
&= \sum_{k=0}^{2^\ell-1}\int_{k/2^\ell}^{(k+1)/2^\ell} \Pi_{\ell,\boldsymbol{c}}(\alpha)\Pi_{\ell(L-1),\boldsymbol{c}}(2^\ell\alpha)\,\mathrm{d}\alpha \\
&= \sum_{k=0}^{2^\ell-1}\frac{1}{2^\ell}\int_0^1 \Pi_{\ell,\boldsymbol{c}}\left(\frac{\tilde\alpha+k}{2^\ell}\right)\Pi_{\ell(L-1),\boldsymbol{c}}(\tilde\alpha+k)\,\mathrm{d}\tilde\alpha \\
&= \int_0^1 \Phi_{\ell,1}(\tilde\alpha)\Pi_{\ell(L-1),\boldsymbol{c}}(\tilde\alpha+k)\,\mathrm{d}\tilde\alpha,
\end{aligned}$$

where we used the transformation $\tilde\alpha = 2^\ell\alpha - k$ in the third and the periodicity of $\Pi_{\ell(L-1),\boldsymbol{c}}$ in the last step, and where we abbreviated

$$\Phi_{\ell,1}(\alpha) = \frac{1}{2^\ell}\sum_{k=0}^{2^\ell-1}\Pi_{\ell,\boldsymbol{c}}\left(\frac{\alpha+k}{2^\ell}\right).$$

This verifies (3.24). Observe that by repeated applications of the identity $\sin(2x) = 2\sin(x)\cos(x)$ we obtain further

$$\frac{|\sin(\alpha\pi)|}{|\cos(\frac{(\alpha+k)\pi}{2^\ell})|} = \frac{|\sin((\alpha+k)\pi)|}{|\cos(\frac{(\alpha+k)\pi}{2^\ell})|} = \frac{2\left|\sin\left(\frac{(\alpha+k)\pi}{2}\right)\right|}{|\cos(\frac{(\alpha+k)\pi}{2^\ell})|} = \ldots = 2^\ell\Pi_{\ell,\boldsymbol{c}}\left(\frac{\alpha+k}{2^\ell}\right).$$

Hence, $\Phi_{\ell,j}$ admits of the recursive representation

$$\Phi_{\ell,j+1}(\alpha) = \frac{1}{2^\ell}\sum_{k=0}^{2^\ell-1} \frac{|\sin(\pi\alpha)|}{2^\ell\left|\cos\left(\frac{(\alpha+k)\pi}{2^\ell}\right)\right|}\Phi_{\ell,j}\left(\frac{\alpha+k}{2^\ell}\right) =: \frac{1}{2^\ell}\sum_{k=0}^{2^\ell-1} g_\ell(\alpha,j,k). \tag{3.25}$$

with initial value $\Phi_{\ell,0} \equiv 1$.

By an inductive argument it is easy to see that $g_\ell(\alpha, j, k) = g_\ell(1-\alpha, j, 2^\ell - 1 - k)$ and, consequently, $\Phi_{\ell,j}(\alpha)$ is symmetric about $\alpha = 1/2$. Furthermore, we define

$$q_{\ell,j}(\alpha) = \frac{\Phi_{\ell,j+1}(\alpha)}{\Phi_{\ell,j}(\alpha)}, \quad M_{\ell,j} = \max_{0\le\alpha\le 1/2} q_{\ell,j}(\alpha), \quad \text{and} \quad m_{\ell,j} = \min_{0\le\alpha\le 1/2} q_{\ell,j}(\alpha).$$

To derive (3.7) we put $\mu(\ell) = M_{\ell,0}$ and hence it remains to prove $M_{\ell,0} = \Phi_{\ell,1}(1/2)$. This follows from the fact that $\Phi_{\ell,1}$ is concave which we establish in the subsequent paragraphs, where we make use of the techniques developed by Fouvry and Mauduit in [23]. For $\ell = 1$ this was shown in [24] and hence we assume $\ell \geq 2$. Furthermore, we already know that

$$\begin{aligned} 2^{2\ell}\Phi_{\ell,1}(\alpha) &= \sum_{k=0}^{2^\ell-1} \frac{\sin(\alpha\pi)}{\left|\cos\left(\frac{(\alpha+k)\pi}{2^\ell}\right)\right|} \\ &= \sum_{k=0}^{2^{\ell-1}-1} \sin(\alpha\pi)\left(\frac{1}{\cos\left(\frac{(\alpha+k)\pi}{2^\ell}\right)} + \frac{1}{\cos\left(\frac{(k+1-\alpha)\pi}{2^\ell}\right)}\right). \end{aligned}$$

For $0 \le u \le 2^{-\ell}$ and $0 \le k < 2^{\ell-1}$ we introduce the functions

$$\psi_k^{(1)}(u) = \frac{\sin(2^\ell u\pi)}{\cos\left(\left(u+\frac{k}{2^\ell}\right)\pi\right)} \qquad \text{and} \qquad \psi_k^{(2)}(u) = \frac{\sin(2^\ell u\pi)}{\cos\left(\left(\frac{k+1}{2^n}-u\right)\pi\right)}.$$

After the change of variable $\alpha = 2^\ell u$ it remains to show that $\sum_{k=0}^{2^{\ell-1}-1}\left(\psi_k^{(1)}(u) + \psi_k^{(2)}(u)\right)$ is concave. It is immediate that

$$\psi_k^{(1)} = \frac{(-1)^k \sin\left(2^\ell\left(u+\frac{k}{2^\ell}\right)\pi\right)}{\cos\left(\left(u+\frac{k}{2^\ell}\right)\pi\right)}, \qquad \text{and} \qquad \psi_k^{(2)} = \frac{(-1)^k \sin\left(2^n\left(\frac{k+1}{2^\ell}-u\right)\pi\right)}{\cos\left(\left(\frac{k+1}{2^\ell}-u\right)\pi\right)}.$$

Using the well-known trigonometric identity $2\sin(x)\cos(y) = \sin(x-y) + \sin(x+y)$ we can inductivley prove that

$$\frac{\sin(2^\ell x)}{\cos(x)} = 2\sum_{l=1}^{2^{\ell-1}}(-1)^l \sin((2l-1)x). \tag{3.26}$$

Indeed,

$$\frac{\sin(4x)}{\cos(x)} = 2\frac{\sin(2x)}{\cos(x)}\cos(2x) = 4\sin(x)\cos(2x) = 2(-\sin(x)+\sin(3x)),$$

and for $\ell \geq 3$ we obtain

$$\begin{aligned}\frac{\sin(2^\ell x)}{\cos(x)} &= 2\frac{\sin(2^{\ell-1}x)}{\cos(x)}\cos(2^{\ell-1}x) = 4\cos(2^{\ell-1}x)\sum_{l=1}^{2^{\ell-2}}(-1)^l\sin((2l-1)x)\\ &= 2\sum_{l=1}^{2^{\ell-2}}(-1)^l\Big(\sin\big(((2l-1)-2^{\ell-1})x\big)+\sin\big(((2l-1)+2^{\ell-1})x\big)\Big)\\ &= 2\sum_{l=1}^{2^{\ell-1}}(-1)^l\sin((2l-1)x),\end{aligned}$$

where we used the induction hypothesis in the second step. Let us focus on $\psi_k^{(1)}$ first. As a consequence of (3.26) we may rewrite

$$\begin{aligned}\sum_{k=0}^{2^{\ell-1}-1}\psi_k^{(1)}(u) &= 2\sum_{l=1}^{2^{\ell-1}}(-1)^l\sum_{k=0}^{2^{\ell-1}-1}(-1)^k\sin\left((2l-1)u\pi + k\frac{2l-1}{2^\ell}\pi\right)\\ &= \sum_{l=1}^{2^{\ell-1}}(-1)^l\sum_{k=0}^{2^{\ell-1}-1}(-1)^k\cos\left((2l-1)u\pi - \pi/2 + k\frac{2l-1}{2^\ell}\pi\right).\end{aligned}$$

We invoke the following formula from [23, p. 345],

$$\sum_{k=0}^{m-1}(-1)^k\cos(a+hk) = \frac{\cos\left(a+\frac{m-1}{2}h+\frac{m-1}{2}\pi\right)\sin\left(\frac{mh}{2}+\frac{m\pi}{2}\right)}{\cos\frac{h}{2}}$$

with $m = 2^{\ell-1}, a = (2l-1)u\pi - \pi/2, h = 2^{-\ell}(2l-1)\pi$ to find that

$$\begin{aligned}&\sum_{k=0}^{2^{\ell-1}-1}\psi_k^{(1)}(u)\\ &= 2\sum_{l=1}^{2^{\ell-1}}(-1)^l\frac{\sin\left((2l-1)u\pi + \frac{(2^{\ell-1}-1)(2l-1)}{2^{\ell+1}}\pi + \frac{2^{\ell-1}-1}{2}\pi\right)\sin\left(\frac{(2l-1)}{4}\pi + \frac{2^{\ell-1}}{2}\pi\right)}{\cos\left(\frac{2l-1}{2^{\ell+1}}\pi\right)}\\ &= 2\sum_{l=1}^{2^{\ell-1}}(-1)^{l+1}\frac{\cos\left((2l-1)\left(u+\frac{2^{n-1}-1}{2^{n+1}}\right)\pi\right)\sin\left(\frac{2l-1}{4}\pi\right)}{\cos\left(\frac{2l-1}{2^{\ell+1}}\pi\right)}.\end{aligned}$$

Observe that the simplification of the numerator in the last line follows a different line of reasoning for $\ell = 2$ than for $\ell \geq 3$, yet the result remains the same. Using $\psi_k^{(2)}(u) = \psi_k^{(1)}(1/2^\ell - u)$ we rewrite

$$\sum_{k=0}^{2^{\ell-1}-1} \psi_k^{(2)}(u) = 2\sum_{l=1}^{2^{\ell-1}} (-1)^{l+1} \frac{\cos\left((2l-1)\left(\frac{2^{\ell-1}+1}{2^{\ell+1}} - u\right)\pi\right) \sin\left(\frac{2l-1}{4}\pi\right)}{\cos\left(\frac{2l-1}{2^{\ell+1}}\pi\right)}.$$

Considering the identity $2\cos(x)\cos(y) = \cos(x+y) + \cos(x-y)$ with $x = (2l-1)\pi/4$ and $y = (2l-1)(u - 2^{-n-1})\pi$ and, subsequently, $\sin((2l-1)\pi/4)\cos((2l-1)\pi/4) = (-1)^{l+1}/2$ we can simplify as follows

$$\sum_{k=0}^{2^{\ell-1}-1} \left(\psi_k^{(1)}(u) + \psi_k^{(2)}(u)\right) = 2\sum_{l=1}^{2^{\ell-1}} \frac{\cos\left((2l-1)\left(u - \frac{1}{2^{\ell+1}}\right)\pi\right)}{\cos\left(\frac{(2l-1)\pi}{2^{\ell+1}}\right)}.$$

Note that $(2l-1)\left(u - \frac{1}{2^{\ell+1}}\right)\pi \in (-\pi/2, \pi/2)$ and $\frac{2l-1}{2^{\ell+1}}\pi \in (0, \pi/2)$. Therefore, each summand is a concave function and, hence, so is $\Phi_{\ell,1}$. This finally proves (3.7).

In order to approach (3.8) we need to show that $(m_{\ell,j})_{j\geq 0}$ is an increasing sequence as well as that $(M_{\ell,j})_{j\geq 0}$ is a decreasing sequence. This can be done as follows, thereby closely following the lines of [2, Proof of Lemma 7]. For each $\alpha \in [0,1]$ we have

$$\begin{aligned} q_{\ell,j}(\alpha) = \frac{\Phi_{\ell,j+1}(\alpha)}{\Phi_{\ell,j}(\alpha)} &= \frac{\sum_{k=0}^{2^\ell-1} \frac{|\sin(\alpha\pi)|}{\left|\cos\left((\alpha+k)\pi/2^\ell\right)\right|} \Phi_{\ell,j}\left(\frac{\alpha+k}{2^\ell}\right)}{\sum_{k=0}^{2^\ell-1} \frac{|\sin(\alpha\pi)|}{\left|\cos\left((\alpha+k)\pi/2^\ell\right)\right|} \Phi_{\ell,j-1}\left(\frac{\alpha+k}{2^\ell}\right)} \\ &\leq \frac{\sum_{k=0}^{2^\ell-1} \frac{|\sin(\alpha\pi)|}{\left|\cos\left((\alpha+k)\pi/2^\ell\right)\right|} \Phi_{\ell,j-1}\left(\frac{\alpha+k}{2^\ell}\right) M_{\ell,j-1}}{\sum_{k=0}^{2^\ell-1} \frac{|\sin(\alpha\pi)|}{\left|\cos\left((\alpha+k)\pi/2^\ell\right)\right|} \Phi_{\ell,j-1}\left(\frac{\alpha+k}{2^\ell}\right)} \\ &= M_{\ell,j-1}, \end{aligned}$$

where we used (3.24) in the second step. Hence, $M_{\ell,j} \leq M_{\ell,j-1}$. In the same spirit it is possible to derive $m_{\ell,j} \geq m_{\ell,j-1}$. Hence, one can define the numbers $\Lambda_1(\ell) = \lim_{j\to\infty} m_{\ell,j}$ and $\Lambda_2(\ell) = \lim_{j\to\infty} M_{\ell,j}$. If we consider additionally

$$\int_0^1 \Pi_{\ell L,\boldsymbol{c}}(\alpha)\,\mathrm{d}\alpha = \int_0^1 \Phi_{\ell,L}(\alpha)\,\mathrm{d}\alpha = \int_0^1 q_{\ell,0}(\alpha)q_{\ell,1}(\alpha)\cdots q_{\ell,L-1}(\alpha)\,\mathrm{d}\alpha$$

it is easy to deduce

$$\Lambda_1(\ell)^{L-k} \int_0^1 \prod_{j=0}^{k-1} q_{\ell,l}(\alpha)\,\mathrm{d}\alpha \leq \int_0^1 \Pi_{\ell L,\boldsymbol{c}}(\alpha)\,\mathrm{d}\alpha \leq \Lambda_2(\ell)^{L-k} \int_0^1 \prod_{j=0}^{k-1} q_{\ell,l}(\alpha)\,\mathrm{d}\alpha$$

for each k and (3.8) follows. This concludes the proof of Proposition 3.16.

It needs to be mentioned that we are in a position to numerically compute the lower and upper bounds for both $\Lambda_1(\ell)$ and $\lambda_2(\ell)$ for small values of ℓ depicted in Figure 3.2 on the basis of the recurrence relation (3.25) as $(M_{\ell,j})_{j\geq 0}$ is a decreasing and $(m_{\ell,j})_{j\geq 0}$ is an increasing sequence. In the case $\ell = 1$ Fouvry and Mauduit ensured that $\Lambda_1(1) = \Lambda_2(1)$, see [24]. As our main interest lies in the exponent of the star discrepancy we settle for the given approximations at the moment and keep a generalization of the more sophisticated result of Fouvry and Mauduit for larger $\ell \in \mathbb{N}$ for future research.

3.2.6 Discussion and open problems

The study of perturbed Halton–Kronecker sequences obviously admits of an entire universe of new problems. Indeed, just consider the simple structure of the generating matrix C, for instance, or the fact that we consider a one-dimensional Kronecker sequence. Hence, only two open problems relevant to this topic are given here which are not obvious from the outset and the big picture is left open.

Nevertheless, it is probably worth mentioning that partial results concerning such *at-hand* generalizations have already been obtained. For instance, with slightly more technical effort it is possible to show the following generalization of Proposition 3.17.

Proposition 3.28. Let p be a prime and let $\boldsymbol{c} = (1, c_1, c_2, \ldots) \in \{0, 1, \ldots, p-1\}$ with infinitely many of the $c_j \neq 0$. We shall denote the one-dimensional digital sequence generated by the infinite matrix $C = \mathrm{Id}$ with its first row replaced with $\boldsymbol{c}$ by $(x_k)_{k\geq 0}$. Furthermore, let $\boldsymbol{y} = (\{\alpha_1 k\}, \{\alpha_2 k\}, \ldots, \{\alpha_d k\})_{k\geq 0}$ be a d-dimensional Kronecker sequence, where we require the numbers $1, \alpha_1, \alpha_2, \ldots, \alpha_d$ to be linearly independent over the rationals. Then the $(d+1)$-dimensional hybrid sequence $(x_k, \boldsymbol{y}_k)_{k\geq 0}$ is subject to

$$D_N^*((x_k, \boldsymbol{y}_k)_{k\geq 0}) \ll_{p,d} \frac{N}{K} + \frac{N}{H} \log N + (\log N)^{d+1} + \left(\frac{3}{2}\right)^d (p-1)$$
$$\times \sum_{l=1}^{\lfloor \log_p K \rfloor} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\} \\ \|\boldsymbol{h}\|_{\ell^\infty} \leq \lfloor H/p^l \rfloor}} \frac{1}{r(\boldsymbol{h})} \left(\frac{1}{\langle p^l\, \boldsymbol{h}\cdot\boldsymbol{\alpha} \rangle} + \sum_{z=1}^{p-1} \sum_{r=0}^{\lfloor \log_p N \rfloor - l} \prod_{j=0}^{r-1} \frac{\left|\sin(p^{j+l+1}\, \boldsymbol{h}\cdot\boldsymbol{\alpha}\, \pi)\right|}{\left|\sin\left((p^{j+l}\, \boldsymbol{h}\cdot\boldsymbol{\alpha} + \frac{z}{p} c_{j+l})\pi\right)\right|} \right)$$

for all $H, K \leq N$ and where $r(h_1, h_2, \ldots, h_d) = \prod_{j=1}^d \max\{1, |h_j|\}$.

As a result of personal communication with Roswitha Hofer this implies, for instance, that choosing $c_j = 1$ for all $j \geq 1$ and $\boldsymbol{\alpha}$ of finite type ς yields in combination with a more sophisticated version of Lemma 3.22 as well as another result by Gel′fond the discrepancy bound

$$D_N^*((x_k, \boldsymbol{y}_k)_{k\geq 0}) \ll_{p,\boldsymbol{\alpha},\epsilon} N^{\max\left\{\Lambda_p, \frac{\varsigma d}{\varsigma d+1}\right\}+\epsilon}$$

for all $\epsilon > 0$, where

$$\Lambda_p = \frac{1}{2}\log_p\left(\frac{p\sin\left(\frac{\pi}{2p}\right)}{\sin\left(\frac{\pi}{2p^2}\right)}\right).$$

For another immediate generalization one could consider one-dimensional Kronecker sequences and periodic $\boldsymbol{c} \in \{0,1\}^{\mathbb{N}}$ again, but allow the periodic blocks of $\boldsymbol{c}$ to be of any form. In this case, this leads to the study of products as given in (3.6). Numeric experiments support a positive answer to the following open problem.

Open Problem 3.29. Is it possible to extend Proposition 3.15 to more general periodic perturbing sequences $\boldsymbol{c}$ over $\{0,1\}$, such that the bounds merely depend on the length of the period and the density of $1'$s in $\boldsymbol{c}$?

Concerning the metric case one might for instance be interested in obtaining better approximations for the numbers $\Lambda_i(\ell)$, $i \in \{1,2\}$, maybe also for larger ℓ. As we have already seen, this task entails maximizing/minimizing the functions $q_{\ell,j}(\alpha)$, $j \geq 0$. Experiments showed that there are always at most two candidates for the extremal point: either $\alpha = 0$ or $\alpha = 1/2$. If this turns out to be true, the overal procedure can be reduced to point evaluations of the functions $\Phi_{\ell,j}(\alpha)$, which would allow us to obtain better approximations even faster.

Open Problem 3.30. Show that approximations of $\Lambda_1(\ell)$ and $\Lambda_2(\ell)$ can be obtained by repeated point evaluations of the functions $\Phi_{\ell,j}(\alpha)$.

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List of Figures

Glossary

Abbreviations

a.c.	admissible and connected (referring to graphs); see Definition 2.38
b.c.f.c.	bounded continued fraction coefficients; see 86
cf.	confer
Ch.	Chapter
e.g.	exempli gratia, for example
etc.	et cetera
f.	following
i.e.	id est, that is
mod.	modulo
p., pp.	page, pages
QMC	quasi-Monte Carlo; see Section 1.2
SBI, SSBI	small ball inequality, signed small ball inequality; see page 12
u.d. mod1	uniformly distributed modulo 1; see Defintion 1.2
w.l.o.g.	without loss of generality
w.r.t.	with respect to

Miscellaneous (unless otherwise defined)

$\approx$	approximately
$\mathcal{A}(\cdot, N, \boldsymbol{x})$, $\mathcal{A}(\cdot, N, J)$	counting part, see page 2 and Definition 1.2
$\mathcal{D}$	class of dyadic intervals, see Definition 2.24
$\mathcal{D}_{\vec{r}}$	special set, see (2.11)
$\mathcal{P}$, $\mathcal{S}$	generic point set, sequence
c^*	star discrepancy constant; see Definition 2.8
$D_N^*(\cdot)$	star discrepancy; see Definition 1.1
$\mathbb{A}_v$	see page 44
$\mathbb{H}_n^d$	d-dimensional hyperbolic vectors, see Definition 2.26
$\mathbb{X}(G)$	special set of hyperbolic vectors, see (2.48)
$\mathbb{X}_1, \mathbb{X}_2$	special sets of hyperbolic vectors, see 46
$\varepsilon^\tau(b)$	see Lemma 2.31
f^*	minimizer of (2.5)
$\Gamma_0, \Gamma_1, \Gamma_2$	special sets, see Definition 2.14
$\mathscr{F}$	space of admissible functions, see Definition 2.14
$\Psi, \Psi_v, \Psi^{\mathrm{sd}}, \Psi^{\neg}$	auxiliary functions, see page 44
ρ	see (2.20)
e	Euler number
i	imaginary unit
σ_0	maximal slope of an admissible function, see Definition 2.14

SP	see page 46
$\tilde{\rho}$	see (2.20)
A_0, A_1, A_2	special partition, see page 14
b	see page 43
d	dimension
$D_N(\cdot,\cdot)$	discrepancy function; see Definition 1.1
$f \circ g$	concatenation of the functions f and g
F_v	auxiliary function, see page 44
I_v	special partition, see (2.21)
N	number of points (size of initial segment of elements) of a point set (a sequence) under consideration
n	integer with $2^{n-2} \leq N < 2^{n-1}$
q	see (2.20)
$Q_{N,\mathcal{P}}(f)$	quasi-Monte Carlo algorithm, see (1.3)
$R_{N,\mathcal{P}}$	integration error; see page 7
$T(G)$	see page 69

Algebra, discrete mathematics and number theory

$\boldsymbol{x} \cdot \boldsymbol{y}$	euclidean dot product of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$
$\boldsymbol{x}^\top$	transpose of the vector $\boldsymbol{x}$
$\mathbb{Z}/b\mathbb{Z}$	additive group of b-adic integers
$\mathcal{C}(V)$	see Definition 2.38

$\mathcal{T}(V)$	generalized tree graphs, see Definition 2.38
$\mathcal{V}(V, l)$	set of partitions of V into l subsets, see Lemma 2.44
Id	identity matrix
$\langle x \rangle$	distance of x to the nearest integer; see page 96
$\lfloor \cdot \rfloor$, $\lceil \cdot \rceil$	floor function, ceiling function
$\{x\}$	fractional part of a non-negative number x
$s_{\boldsymbol{\gamma}}$	weighted sum of digits in base 2 with weight sequence $\boldsymbol{\gamma}$, see page 98
$\binom{W}{k}$	set of all subsets of W with k elements

Set theory

$[0, \boldsymbol{x})$	the box $[0, x_1) \times [0, x_2) \times \cdots \times [0, x_d)$ for $\boldsymbol{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$
$\#A$	cardinality of A
$\mathbb{N}$, $\mathbb{N}_0$	positive integers, non-negative integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{Z}$	integers
$\emptyset$	empty set
$[\![k]\!]$	the set $\{1, 2, \ldots, k\}$ for positive integers k
$A \cap B$	intersection of A with B
$A \cup B$	union of A and B
$A \setminus B$	set theoretic difference of A and B

$A \times B$	cartesian product of A and B
A^d	d-fold cartesian product of A with itself

Analysis, measure theory, probability and topology

$\boldsymbol{\gamma}^{(\ell)}$	shifted sequence obtained by ℓ shifts of the original sequence $\boldsymbol{\gamma}$
$\mathbb{E}$	expectation, conditional expectation; see page 39
$\mathbb{1}_A$	indicator function of the set A
$\mathbb{P}$	probability; see page 39
exp	exponential function
λ_d	d-dimensional Lebesgue measure
$\langle \cdot, \cdot \rangle$	inner product in L^2
$\lim_{x \downarrow x_0}, \lim_{x \uparrow x_0}$	right limit, left limit
$\ll_X, \gg_X, \asymp_X$	see page 86
$\log_b(x)$	logarithm of x in base b, index omitted if $b = \mathrm{e}$
$\lesssim, \gtrsim, \simeq$	see page 39
$\lVert\cdot\rVert_{\ell^p}$	stanard norm in the sequence space ℓ^p
$\Pi_{r,\boldsymbol{\gamma}}$	lacunary trigonometric product, see (3.6)
$\mathscr{S}$	square function, see Proposition 2.30
sgn	sign function
$f_{\vec{r}}$	Rademacher function, see Definition 2.26
h_R	Haar function, see Definition 2.24

$I_d(f)$	integral operator, see (1.3)
L^p	Banach space of p-integrable functions on $[0,1)^d$ with respect to the Lebesgue measure, $1 \leq p \leq \infty$
$\|\cdot\|_p$	norm in the space L^p