JOHANNES KEPLER
UNIVERSITÄT LINZ | JKU

TNF

Technisch-Naturwissenschaftliche
Fakultät

Quasi-Monte Carlo integration of generalized Walsh series based on digital nets and polynomial lattices

MASTERARBEIT

zur Erlangung des akademischen Grades

Diplomingenieur

im Masterstudium

Mathematik in den Naturwissenschaften

Eingereicht von:
Florian Puchhammer

Angefertigt am:
Institut für Finanzmathematik

Beurteilung:
Friedrich Pillichshammer

Linz, Jänner, 2013

Abstract

This thesis is aimed at tackling the multivariate integration problem (with respect to quasi-Monte Carlo-rules) in the space of generalized Walsh series. To this end, the family of generalized Walsh functions defined over the finite field $\mathbb{F}_q$, where q is a prime power, is introduced and some of their most important properties are presented (see also [16], for instance, or [6] for a survey on the integer-base case). Furhtermore, after having outlined the basic principles of reproducing kernel Hilbert spaces (see, e.g.,[1]), the information that is hereby elaborated is deployed to construct the weighted Hilbert space of generalized Walsh series over $\mathbb{F}_q$, i.e. $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$, and to find its reproducing kernel $K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$, which, as is shown in the course of this thesis, may be simplified such that it can be evaluated computationally, (see [6] for the prime case). This fact immediately comes into play when analizing the error behavior of quasi-Monte Carlo-rules applied in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$, since the worst-case error strongly depends on $K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ (cf. [5, 6]). In particular, the employment of digital (t, m, s)-nets (see [6], for instance) as sample points for quasi-Monte Carlo integration is investigated, which helps to relate the worst-case error to the so-called dual net and thereby reduces the computational cost and, above all, to provide existence results for “good” sample points (see, e.g., [6] for the prime case). Besides, also (strong) tractability of integration in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ is taken into consideration to determine the quality of quasi-Monte Carlo integration in this space resulting in necessary as well as sufficient conditions on the sequence of weights $\boldsymbol{\gamma} = (\gamma_j)_{j\in\mathbb{N}}$ (see [6] again, for instance). In order to move towards concrete point sets, four construction algorithms for digital nets over $\mathbb{F}_q$ are presented. These include the component-by-component construction, a Korobov type construction, the construction method by Niederreiter and the construction method by Sobol‘, where the first two are examples for generating so-called polynomial lattices (see also [4, 5]). Moreover, it is shown that all of these algorithms (except for the Korobov type construction, which has its advantage in the complexity of its construction) satisfy an estimate for the worst-case error which is independent of the dimension s under certain conditions and are hence candidates for exploiting tractability and strong tractability of integration in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$. For quasi-Monte Carlo-rules using point sets obtained by the Korobov type construction it is still possible (again, under certain conditions) to bound the worst-case error with a polynomial dependence on the dimension and thus tractability applies, nevertheless, (see [3, 4] for the prime case).

Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Diplomarbeit selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Die vorliegende Diplomarbeit ist mit dem elektronisch übermittelten Textdokument identisch.

Contents

1 Introduction

We are primarily interested in approximating the integral of a function f over the s-dimensional unit cube $[0,1)^s$ by using an equally weighted n-point quadrature rule, i.e.

$$\int_{[0,1)^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \approx \frac{1}{n}\sum_{h=0}^{n-1} f(\mathbf{x}_h),$$

where $\mathbf{x}_0,\dots,\mathbf{x}_{n-1}$ are deterministically chosen sample points. Rules of the above form are usually referred to as quasi-Monte Carlo-rules (QMC-rules). Note that, in the above expression, the left handside equals the mean of the function f over the unit cube. Thus, it is a reasonable approach to use the arithmetic mean of f with respect to the sample points chosen as an approximation, (cf. [6, p. 16]).

General theory on this topic confirms the self-suggesting suspicion that the error behavior of multivariate integration by means of QMC-rules largely depends on the choice of sample points as well as on which functions are to be approximated (see [5, Remark 2.19], for instance). Hence, in this thesis we will clearly set out which function space and which kind of point sets we will restrict ourselves to.

To this end we introduce so-called *generalized Walsh functions* over the finite field $\mathbb{F}_q$, where q denotes a prime power. These are, generally speaking, a family of special step functions in whose definition finite fields play an essential role. Finally, our working space $\mathscr{H}_{\mathrm{wal},s,\beta,\gamma}$ comprises *generalized Walsh series*, i.e. series of generalized Walsh functions equipped with complex coefficients, with finite norm (the respective norm will be introduced in Section 2.4).

As it turns out, $\mathscr{H}_{\mathrm{wal},s,\beta,\gamma}$ is a *reproducing kernel Hilbert space*. These may be briefly described as Hilbert spaces of functions for which there exists a bivariate function K with the properties that K is contained in the Hilbert space whenever we fix one variable and, furthermore, in some sense it represents any other function in this space via the inner product (see Definition 2.1 for a full explanation).

Secondly, the point sets we intend to employ are so-called *digital* (t,m,s)-*nets* over the finite field $\mathbb{F}_q$. (t,m,s) nets were originally introduced by H. Niederreiter in [13]. These are special point sets consisting of q^m points in $[0,1)^s$ for which it is known that the parameter t indicates how well the points are distributed in $[0,1)^s$ (cf. [5, Chapter 5.5.1], for instance). Now, digital (t,m,s)-nets over $\mathbb{F}_q$ are (t,m,s)-nets which are constructed from a choice of s freely selectable $m \times m$ matrices over $\mathbb{F}_q$ and which constitute the most widely used construction scheme for (t,m,s)-nets in practical applications, (cf. [6, p. 158]).

For determining the quality of applying digital nets in QMC-integration in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ with respect to the speed of convergence we are mainly interested in two things, namely the *worst-case error* and *(strong) tractability*. The first term hereby indicates how large the actual error, i.e.

$$\left| \int_{[0,1)^s} f(\mathbf{x}) \, \mathrm{d}\mathbf{x} - \frac{1}{n} \sum_{h=1}^{n} f(\mathbf{x}_h) \right| ,$$

can get for f taken from the closed unit ball of $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ and for given sample points $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Whereas the term tractability so to speak states whether there exists a QMC-rule such that the minimal number of points necessary to attain a certain error bound $\epsilon \in (0,1)$ is of magnitude order $s^b \epsilon^{-a}$ for every $s \in \mathbb{N}$, $a, b \geqslant 0$. If $b = 0$ we speak of *strong tractability*.

So, consequently, in this thesis it is investigated, how large the worst-case error actually is and whether, or, more precisely, under which conditions integration in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ is (strongly) tractable and above all, if there exist digital nets such that (strong) tractability can be exploited. It is to mention that finding the answer to these questions is vastly faciliated by exploiting the fact that $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ is a reproducing kernel Hilbert space.

Moreover, since the study of the above paragraph merely provides existence results, the fifth chapter of this thesis is dedicated to present four well-known algorithms to construct digital nets over $\mathbb{F}_q$ – namely the construction method by Niederreiter, the construction method by Sobol‘ and another two for constructing polynomial lattices – which fulfill the above properties.

As a matter of fact, much of the theory given in this thesis has already been established in [3, 4, 5, 6] for the case where q is a prime number. Therefore, the integral part of this work is to extend the definitions given therein, to adapt the results accordingly and to adjust or renew the proofs of these results. Furthermore, the fact that $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ actually is a (reproducing kernel) Hilbert space shall be demonstrated in a clear and precise way.

2 The reproducing kernel Hilbert space $\mathscr{H}_{\mathrm{wal},s,\beta,\gamma}$

2.1 General theory on reproducing kernel Hilbert spaces

This section is dedicated to briefly discuss *reproducing kernels* and *reproducing kernel Hilbert spaces* and some of their basic properties. First off, we need to define the respective terms.

Definition 2.1 (Reproducing kernel Hilbert space)**.** Let $\mathscr{H}$ be a Hilbert space of functions $f : X \to \mathbb{C}$, where X is a given set. Let the inner product on $\mathscr{H}$ be denoted by $\langle \cdot\, , \cdot \rangle$. Then $\mathscr{H}$ is a *reproducing kernel Hilbert space* iff there exists a function $K : X \times X \to \mathbb{C}$ for which the following two properties hold:

(RK1) $\forall y \in X: \; K(\cdot, y) \in \mathscr{H}$ and

(RK2) $\forall y \in X \; \forall f \in \mathscr{H}: \; f(y) = \langle f, K(\cdot, y)\rangle$,

where $K(\cdot, y)$ is viewed at as a function in the first variable and also the inner product is taken with respect to the first variable.

A function K satisfying the above properties is referred to as a *reproducing kernel for $\mathscr{H}$*. Additionally, it should be mentioned that **(RK2)** in the above definition is called *reproducing property.*

(cf. [5, Definition 2.5])

Certainly, a comprehensive theory has evolved around reproducing kernel Hilbert spaces (see [1], for instance). In order to let the reader become more familiar with this concept the proposition below summarizes some of their basic properties.

Proposition 2.2. *Let X be a set. Then the following holds:*

(i) Let $\mathscr{H}$ be a Hilbert space of functions $f : X \to \mathbb{C}$. Then a reproducing kernel for $\mathscr{H}$ exists if and only if the linear functional

$$\begin{array}{rccc} T_y: & \mathscr{H} & \longrightarrow & \mathbb{C} \\ & f & \longmapsto & f(y) \end{array} \tag{1}$$

is bounded for every $y \in X$ (cf. [1, p. 343, item 2]).

Furthermore, a function $K\colon X{\times}X \to \mathbb{C}$ *which fulfills* ***(RK1)*** *and* ***(RK2)*** *is*

(ii) symmetric, i.e.

$$\forall x, y \in X\colon \quad K(x,y) = \overline{K(y,x)},$$

(iii) positive semi-definite, which means that

$$\forall n \in \mathbb{N} \quad \forall a_0, \ldots, a_{n-1} \in \mathbb{C} \quad \forall x_0, \ldots, x_{n-1} \in X$$

it holds that

$$\sum_{i,j=0}^{n-1} \overline{a}_i a_j K(x_i, x_j) \geqslant 0;$$

(iv) and unique (cf. [5, p. 22, items P3, P5, P4 and Remark 2.7]).

Proof. (Item (i) taken from [1, pp. 343f.], items (ii)-(iv) adapted from [5, p. 22]).

Let X be a set.

(i) Let $\mathscr{H}$ be as stated above and T_y be defined as in (1) for an arbitrary $y \in X$. Furthermore, let K be a reproducing kernel for $\mathscr{H}$. By using the reproducing property of K in the second step for f one obtains

$$\begin{aligned} |T_y f| = |f(y)| &= |\langle f, K(\cdot, y)\rangle| \\ &\leqslant \|f\| \|K(\cdot, y)\| \end{aligned}$$

as a consequence of the Cauchy-Schwarz inequality. Since – due to **(RK1)** – the function $K(\cdot, y)$ is in $\mathscr{H}$, we may deduce that T_y is bounded.

Conversely, assume that T_y defined by $T_y f = f(y)$ is a bounded functional for every $y \in X$. Then, from Riesz' representation theorem it follows that there exists a function $k_y \in \mathscr{H}$ such that

$$f(y) = \langle f, k_y \rangle$$

for every $f \in \mathscr{H}$. Clearly, $K(x,y) := k_y(x)$ is a reproducing kernel for $\mathscr{H}$.

Now, let $K\colon X \times X \to \mathbb{C}$ be a function satisfying **(RK1)** and **(RK2)**.

(ii) The symmetry of $K(x,y)$ follows from the fact that

$$K(x,y) = \langle K(\cdot,y), K(\cdot,x)\rangle = \overline{\langle K(\cdot,x), K(\cdot,y)\rangle} = \overline{K(y,x)}.$$

(iii) Let $n \in \mathbb{N}$. Then, for any choice of $a_0,\dots,a_{n-1} \in \mathbb{C}$ and for every $x_0,\dots,x_{n-1} \in X$ it holds that

$$\begin{aligned}\sum_{i,j=0}^{n-1} \overline{a}_i a_j K(x_i,x_j) &= \sum_{i,j=0}^{n-1} \overline{a}_i a_j \langle K(\cdot,x_j), K(\cdot,x_i))\rangle \\ &= \left\langle \sum_{j=0}^{n-1} a_j K(\cdot,x_j), \sum_{i=0}^{n-1} a_i K(\cdot,x_i) \right\rangle \\ &= \left\| \sum_{i=0}^{n-1} a_i K(\cdot,x_i) \right\|^2 \geqslant 0,\end{aligned}$$

which proves the statement.

(iv) For the uniqueness part assume that there exist two functions K and $\tilde{K}$ mapping from $X \times X$ onto $\mathbb{C}$ which satisfy **(RK1)** and **(RK2)** and therefore also (ii). Then,

$$\tilde{K}(x,y) = \langle \tilde{K}(\cdot,y), K(\cdot,x)\rangle = \overline{\langle K(\cdot,x), \tilde{K}(\cdot,y)\rangle} = \overline{K(y,x)} = K(x,y)$$

for every $x,y \in X$. □

Later, in Section 4, the concept of reproducing kernel Hilbert spaces will be used to obtain error bounds for Quasi-Monte Carlo-rules (QMC-rules). As these entail evaluating a function f at previously chosen sample points, it seems to be a reasonable prerequisite that the functional T_y from Proposition 2.2.(i) is continuous (cf. [5, p. 25]) and this is, as we have just seen, equivalent to the existence of a reproducing kernel.

Another interesting fact (among many others) is given by N. Aronszajn in [1, p. 344]. It states that any bivariate function fulfilling items (ii) and (iii) from Proposition 2.2, i.e. symmetry and positive semi-definiteness, already uniquely determines a reproducing kernel Hilbert space and its inner product. This justifies the usage of *"reproducing kernel"* as a stand-alone term.

2.2 Generalized Walsh functions over a finite field

Generalized Walsh functions may appear in many different forms. For instance in [6, Definitions 1, 2] one can find a definition of such using a general integer base $b \geqslant 2$, or, as G. Pirsic did in [15, Definition 9], they can also be defined over groups. The aim of this chapter is to introduce Walsh functions over the finite field $\mathbb{F}_q$, where q is a prime power. For the following introductory part we closely follow [16, p. 388].

In all of what follows let $q = p^r$, where p is a prime number and $r \in \mathbb{N}$. Furthermore, for every positive integer b we denote by $\mathbb{Z}_b$ the residue class ring modulo b, which we will usually identify with the least residue system modulo b, i.e. $\{0, 1, \ldots, b-1\}$. Additionally, let $\varphi_1 \colon \mathbb{Z}_q \to \mathbb{F}_q$ be a bijection with $\varphi_1(0) = 0$, i.e. the zero element of $\mathbb{Z}_q$ is mapped onto the zero element of $\mathbb{F}_q$. It follows from general theory of finite fields that there exists an isomorphism between the additive groups $\mathbb{F}_q$ and $\mathbb{Z}_p^r$, name it ψ. By setting $\eta := \psi \circ \varphi_1$ one obtains the commutative diagram given in Figure 1.

$$\begin{tikzcd}
\mathbb{Z}_q \arrow[r, "\varphi_1"] \arrow[dr, "\eta"'] & \mathbb{F}_q \arrow[d, "\psi"] \\
& \mathbb{Z}_p^r
\end{tikzcd}$$

Figure 1: Commutative diagram, (cf. [16, Definition 2.3]).

This leads to the definition of those generalized Walsh functions which will be investigated in this thesis.

Definition 2.3 (Generalized Walsh functions)**.** First, we consider the one-dimensional case. To this end, let $q = p^r$, φ_1, ψ and η be as described in the paragraph above. Additionally, let $k \in \mathbb{N}_0$ have the base q representation $k = \kappa_1 + \kappa_2 q + \cdots + \kappa_m q^{m-1}$ where $\kappa_j \in \mathbb{Z}_q$ for all $1 \leqslant j \leqslant m$.

Furthermore, identify $x \in [0, 1)$, too, with its base q representation, i.e. $x = x_1 q^{-1} + x_2 q^{-2} + \cdots$. For reasons of uniqueness of this representation it is demanded, that for any natural j there exists an index $j_0 \geqslant j$

such that x_{j_0} is different from $q-1$.

Then we call the function

$$\begin{aligned} {}_{\mathbb{F}_q,\varphi_1}\text{wal}_k : [0,1) &\longrightarrow \mathbb{C} \\ x &\longmapsto \prod_{l=1}^{m} \exp\left(\frac{2\pi \mathrm{i}}{p}\, \eta(\kappa_l) \cdot \eta(x_l)\right), \end{aligned}$$

the one-dimensional kth generalized Walsh function over $\mathbb{F}_q$ with respect to φ_1, where " i" denotes the imaginary unit and "·" stands for the Euklidean product

Now, the s-dimensional case can be constructed from the above one. For this reason let $s \geqslant 2$, $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$ and $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$. The respective multivariate generalized Walsh function is then defined as ${}_{\mathbb{F}_q,\varphi_1}\text{wal}_{\mathbf{k}} \colon [0,1)^s \to \mathbb{C}$,

$${}_{\mathbb{F}_q,\varphi_1}\text{wal}_{\mathbf{k}}(\mathbf{x}) := \prod_{j=1}^{s} {}_{\mathbb{F}_q,\varphi_1}\text{wal}_{k_j}(x_j).$$

(cf. [16], Definition 2.3)

To avoid tedious notation the subscripts $\mathbb{F}_q$ and φ_1 will be omitted from now on, unless they are required to overcome ambiguities. So, simply the abbreviation wal_k (or $\text{wal}_{\mathbf{k}}$ respectively) will be used. Also, in what follows the term "generalized" will be dropped most of the time.

2.3 Basic properties of generalized Walsh functions over $\mathbb{F}_q$

In this section we aim at gathering important information on Walsh functions defined over a finite field, some of which will be of essential use later. For a better understanding it is necessary to point out that the variables $q = p^r$ as well as the mappings φ_1, ψ and $\eta = \psi \circ \varphi_1$ have already been arbitrarily chosen or defined in the first paragraph of Section 2.2.

First of all, we introduce two binary operations $\oplus_{\varphi_1}$ and $\ominus_{\varphi_1}$.

Definition 2.4. Let $x = \sum_{i=w}^{\infty} x_i q^{-i}$ and $y = \sum_{i=w}^{\infty} y_i q^{-i}$, $w \in \mathbb{Z}$. Then,

$$x \oplus_{\varphi_1} y := \sum_{i=w}^{\infty} a_i q^{-i}, \quad \text{where} \quad a_i = \varphi_1^{-1}(\varphi_1(x_i) + \varphi_1(y_i)),$$
$$x \ominus_{\varphi_1} y := \sum_{i=w}^{\infty} b_i q^{-i}, \quad \text{where} \quad b_i = \varphi_1^{-1}(\varphi_1(x_i) - \varphi_1(y_i)).$$

Additionally, $\ominus_{\varphi_1} x$ is set as $0 \ominus_{\varphi_1} x$ and if $\mathbf{x}$ and $\mathbf{y}$ are vectors of the same dimension the above operations are understood as being taken componentwise.

In accordance to Definition 2.3 it is required that the sequences a_i and b_i as given above do not possess infinitely many consecutive elements equal to $q-1$, otherwise we consider the operation not defined.

(cf. [16, p. 388])

Again, one should keep in mind that the operations $\oplus_{\varphi_1}$ and $\ominus_{\varphi_1}$ as defined above depend on φ_1. As φ_1 is considered being arbitrarily chosen, however, the more convenient notation $\oplus$ and $\ominus$ will be used from now on.

The following theorem provides a close connection between the product of Walsh functions and the binary operations from above.

Theorem 2.5. *For all k, $l \in \mathbb{N}_0$ and all x, $y \in [0,1)$ it holds that*

$$\begin{aligned} \mathrm{wal}_k(x)\,\mathrm{wal}_l(x) &= \mathrm{wal}_{k\oplus l}(x) &\quad \textit{and} \quad \mathrm{wal}_k(x)\overline{\mathrm{wal}_l(x)} &= \mathrm{wal}_{k\ominus l}(x), \\ \mathrm{wal}_k(x)\,\mathrm{wal}_k(y) &= \mathrm{wal}_k(x\oplus y) &\quad \textit{and} \quad \mathrm{wal}_k(x)\overline{\mathrm{wal}_k(y)} &= \mathrm{wal}_k(x\ominus y), \end{aligned}$$

wherever $x \oplus y$ and $x \ominus y$ respectively is defined.

(cf. [16, Proposition 2.4, item 1])

Proof. Let $k = \kappa_1 + \cdots + \kappa_m q^{m-1}$, $l = \lambda_1 + \cdots + \lambda_m q^{m-1}$ and $x = x_1 q^{-1} + x_2 q^{-2} + \cdots$ be the base q representations of k, l and x. Then it follows from the facts that ψ is an isomorphism and $\psi = \eta \circ \varphi_1^{-1}$ that

$$\begin{aligned}
\mathrm{wal}_k(x)\,\mathrm{wal}_l(x) &= \prod_{j=1}^{m} \exp\left(\frac{2\pi\mathrm{i}}{p}\,\eta(x_j)\cdot\big(\eta(\kappa_j)+\eta(\lambda_j)\big)\right)\\
&= \prod_{j=1}^{m} \exp\left(\frac{2\pi\mathrm{i}}{p}\,\eta(x_j)\cdot\psi\big(\varphi_1(\kappa_j)+\varphi_1(\lambda_j)\big)\right)\\
&= \prod_{j=1}^{m} \exp\left(\frac{2\pi\mathrm{i}}{p}\,\eta(x_j)\cdot\eta\circ\varphi_1^{-1}\big(\varphi_1(\kappa_j)+\varphi_1(\lambda_j)\big)\right)\\
&= \mathrm{wal}_{k\oplus l}(x).
\end{aligned}$$

The proofs of the other identities follow exactly the same pattern. It is only left to mention that complex conjugation yields a minus in the exponent and therefore a "$\ominus$" is obtained in the end. □

If one takes a closer look on the definition of Walsh functions over $\mathbb{F}_q$ one might notice that these are step functions, which is indeed the case, as the following lemma shows.

Lemma 2.6. *Let $k \in \mathbb{N}$ and m be a positive integer such that $q^{m-1} \leqslant k < q^m$. Then the restriction of wal_k to an interval of the form $[a/q^m,(a+1)/q^m) \subseteq [0,1)$ is $\mathrm{wal}_k(a/q^m)$ identically. Furthermore, $\mathrm{wal}_0 \equiv 1$.*

(cf. [5, Proposition A.2])

Proof. Here, we use the same approach as in [5, pp. 559f.]. Since $q^{m-1} \leqslant k < q^m$ the q-adic expansion of k is of the form $k = \kappa_1 + \kappa_2 q + \cdots + \kappa_m q^{m-1}$. Let $a = \alpha_1 + \alpha_2 q + \cdots + \alpha_m q^{m-1}$ be the q-adic expansion of an integer a, $0 \leqslant a < q^m$. Thus, $J = [a/q^m,(a+1)/q^m)$ is contained in $[0,1)$.

Any $x \in J$ possesses a q-adic expansion of the form

$$x = \alpha_m q^{-1} + \cdots + \alpha_1 q^{-m} + \xi_{m+1} q^{-(m+1)} + \xi_{m+2} q^{-(m+2)} + \cdots$$

with suitable digits ξ_j, $0 \leqslant \xi_j \leqslant q-1$, $j \geqslant m+1$. We notice that the first m summands are the same as those in the q-adic expansion of a/q^m. We now consider this observation in our definition of Walsh functions (Definition 2.3) and obtain

$$\mathrm{wal}_k(x) = \prod_{l=1}^{m} \exp\left(\frac{2\pi \mathrm{i}}{p}\ \eta(\kappa_l) \cdot \eta(\alpha_{m+1-l})\right) = \mathrm{wal}_k\left(\frac{a}{q^m}\right).$$

If $k = 0$ we use the fact that $\varphi_1(0) = 0$ in $\mathbb{F}_q$ and, since ψ is an isomorphism, we have $\psi(0) = \mathbf{0}$, the zero element in $\mathbb{Z}_p^r$, and thus $\eta(0) = \mathbf{0}$. From this one sees immediately that for any $x \in [0,1)$ it holds that $\mathrm{wal}_0(x) = 1$. □

By exploiting the result of this lemma we obtain another interesting property of Walsh functions over $\mathbb{F}_q$.

Lemma 2.7. *We have*

$$\int_0^1 \mathrm{wal}_0(x)\,\mathrm{d}x = 1 \qquad \textit{and} \qquad \int_0^1 \mathrm{wal}_k(x)\,\mathrm{d}x = 0 \quad \textit{if } k \in \mathbb{N}.$$

(cf. [16, Proposition 2.4, item 3])

Proof. The first identity follows immediately from Lemma 2.6, as $\mathrm{wal}_0(x) = 1$ for any $x \in [0,1)$.

Now, assume that $k \in \mathbb{N}$ with q-adic expansion $\kappa_1 + \kappa_2 q + \cdots + \kappa_m q^{m-1}$. Then, as it was also done in the proof of [5, Proposition A.9], by applying Lemma 2.6 the integral can be rewritten in the following way:

$$\int_0^1 \mathrm{wal}_k(x)\,\mathrm{d}x = \sum_{a=0}^{q^m-1} \int_{\frac{a}{q^m}}^{\frac{a+1}{q^m}} \mathrm{wal}_k(x)\,\mathrm{d}x = \frac{1}{q^m} \sum_{a=0}^{q^m-1} \mathrm{wal}_k\left(\frac{a}{q^m}\right).$$

This equals zero, for if $0 \leqslant a < q^m$ with q-adic expansion

$$a = \alpha_1 + \alpha_2 q + \cdots + \alpha_m q^{m-1},$$

then a/q^m has the q-adic expansion

$$\frac{a}{q^m} = \alpha_m q^{-1} + \cdots + \alpha_1 q^{-m} =: a_1 q^{-1} + \cdots + a_m q^{-m}.$$

Therefore we can adapt the proof of [5, Lemma A.8] to our purposes and obtain

$$\begin{aligned}\sum_{a=0}^{q^m-1} \mathrm{wal}_k\left(\frac{a}{q^m}\right) &= \sum_{a=0}^{q^m-1} \exp\left(\frac{2\pi \mathrm{i}}{p}\,\eta(\kappa_1)\cdot\eta(a_1)\right)\cdots\exp\left(\frac{2\pi \mathrm{i}}{p}\,\eta(\kappa_m)\cdot\eta(a_m)\right)\\ &= \prod_{l=1}^{m}\left(\sum_{j=0}^{q-1}\exp\left(\frac{2\pi \mathrm{i}}{p}\,\eta(\kappa_l)\cdot\eta(j)\right)\right)\\ &= \prod_{l=1}^{m}\left(\sum_{\mathbf{a}\in\mathbb{Z}_p^r}\exp\left(\frac{2\pi \mathrm{i}}{p}\,\eta(\kappa_l)\cdot\mathbf{a}\right)\right),\end{aligned}$$

since $q = p^r$ and $\eta\colon \mathbb{Z}_q \to \mathbb{Z}_p^r$ is bijective.

For $1 \leqslant l \leqslant m$ and $1 \leqslant j \leqslant r$ we denote by $\eta^{(j)}(\kappa_l)$ the jth component of the vector $\eta(\kappa_l)$, i.e.

$$\eta(\kappa_l) := \left(\eta^{(1)}(\kappa_l), \ldots, \eta^{(r)}(\kappa_l)\right)^\top,$$

where $\mathbf{y}^\top$ denotes the transpose of a vector $\mathbf{y}$.

Now we can continue as follows:

$$\prod_{l=1}^{m}\left(\sum_{\mathbf{a}\in\mathbb{Z}_p^r}\exp\left(\frac{2\pi \mathrm{i}}{p}\,\eta(\kappa_l)\cdot\mathbf{a}\right)\right) = \prod_{l=1}^{m}\prod_{j=1}^{r}\left(\sum_{a=0}^{p-1}\exp\left(\frac{2\pi \mathrm{i}}{p}\,\eta^{(j)}(\kappa_l)a\right)\right).$$

As $k > 0$ we know that there exists a pair of integers (l_0, j_0), $1 \leqslant l_0 \leqslant m$ and $1 \leqslant j_0 \leqslant r$, such that $\eta^{(j_0)}(\kappa_{l_0}) \neq 0$. For such a pair the sum in the above expression is a geometric sum which simplifies to

$$\sum_{a=0}^{p-1}\exp\left(\frac{2\pi \mathrm{i}}{p}\,\eta^{(j_0)}(\kappa_{l_0})a\right) = \frac{1-\exp\left(2\pi \mathrm{i}\;\eta^{(j_0)}(\kappa_{l_0})\right)}{1-\exp\left(\frac{2\pi \mathrm{i}}{p}\;\eta^{(j_0)}(\kappa_{l_0})\right)} = 0.$$

Thus, the whole double-product equals zero and hence

$$\int_0^1 \mathrm{wal}_k(x)\,\mathrm{d}x = 0$$

for $k \in \mathbb{N}$. □

With this knowledge it is now easy to prove the following orthogonality properties.

Theorem 2.8. *For any $k,l \in \mathbb{N}_0$ we have*

$$\int_0^1 \mathrm{wal}_k(x)\overline{\mathrm{wal}_l(x)}\,\mathrm{d}x = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{else.} \end{cases}$$

(cf. [16, Proposition 2.4, item 4])

Proof. (Adapted from [5, Proposition A.10].)

From Theorem 2.5 we know that

$$\int_0^1 \mathrm{wal}_k(x)\overline{\mathrm{wal}_l(x)}\,\mathrm{d}x = \int_0^1 \mathrm{wal}_{k\ominus l}(x)\,\mathrm{d}x.$$

Now, since φ_1 is bijective and $\varphi_1(0) = 0$ in $\mathbb{F}_q$ it follows that

$$k \ominus l = 0 \quad \Longleftrightarrow \quad k = l.$$

Applying Lemma 2.7 completes the proof. □

Another very important result has been shown by G. Pirsic in [15, Satz 3], stating that the system of Walsh functions over groups (as defined therein) is dense in $L_2([0,1)^s)$ for any dimension $s \geqslant 1$. As a special case of this result together with Theorem 2.8 we obtain:

Theorem 2.9. *Let $s \geqslant 1$ be an integer. Then $\{\mathrm{wal}_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}_0^s\}$ is a complete orthonormal system in $L_2([0,1)^s)$.*

Proof. Can be found in [15, Satz 3]. □

Note that, due to this result, we can assign any function f which is square integrable on $[0,1)^s$ to a series of the form

$$f(\mathbf{x}) \sim \sum_{\mathbf{k}\in\mathbb{N}_0^s} \hat{f}(\mathbf{k})\,\mathrm{wal}_{\mathbf{k}}(\mathbf{x}), \tag{2}$$

where

$$\hat{f}(\mathbf{k}) := \int_{[0,1)^s} f(\mathbf{x})\overline{\mathrm{wal}_{\mathbf{k}}(\mathbf{x})}\,\mathrm{d}\mathbf{x}. \tag{3}$$

Later, in Section 2.4 we will call a series as given in (2) a *generalized Walsh series*, where the coefficient $\hat{f}(\mathbf{k})$ will be referred to as the $\mathbf{k}$th *Walsh-Fourier coefficient* (see [6, p. 154], for instance).

Moreover, according to [5, Theorem A.20], we can even obtain equality in (2) if f is continuous and

$$\sum_{\mathbf{k}\in\mathbb{N}_0^s} \left|\hat{f}(\mathbf{k})\right| < \infty.$$

To provide the reader with a better overview, the next proposition summarizes and generalizes the results shown so far for Walsh functions over $\mathbb{F}_q$.

Proposition 2.10. *Let $s \geqslant 1$ be an integer.*

(i) For all $\mathbf{k}$, $\mathbf{l} \in \mathbb{N}_0^s$ and for all $\mathbf{x}$, $\mathbf{y} \in [0,1)^s$ the following identities hold:

$$\mathrm{wal}_{\mathbf{k}}(\mathbf{x})\,\mathrm{wal}_{\mathbf{l}}(\mathbf{x}) = \mathrm{wal}_{\mathbf{k}\oplus\mathbf{l}}(\mathbf{x}) \quad \text{and} \quad \mathrm{wal}_{\mathbf{k}}(\mathbf{x})\overline{\mathrm{wal}_{\mathbf{l}}(\mathbf{x})} = \mathrm{wal}_{\mathbf{k}\ominus\mathbf{l}}(\mathbf{x}),$$
$$\mathrm{wal}_{\mathbf{k}}(\mathbf{x})\,\mathrm{wal}_{\mathbf{k}}(\mathbf{y}) = \mathrm{wal}_{\mathbf{k}}(\mathbf{x}\oplus\mathbf{y}) \quad \text{and} \quad \mathrm{wal}_{\mathbf{k}}(\mathbf{x})\overline{\mathrm{wal}_{\mathbf{k}}(\mathbf{y})} = \mathrm{wal}_{\mathbf{k}}(\mathbf{x}\ominus\mathbf{y}),$$

provided that $\mathbf{x}\oplus\mathbf{y}$ and $\mathbf{x}\ominus\mathbf{y}$ are defined.

(ii) Denote by $\mathbf{0} = (0,\ldots,0)^\top$ the s-dimensional zero vector. Then:

$$\int_{[0,1)^s} \mathrm{wal}_{\mathbf{0}}(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1 \qquad \text{and} \qquad \int_{[0,1)^s} \mathrm{wal}_{\mathbf{k}}(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \quad \text{if } \mathbf{k}\in\mathbb{N}^s.$$

(iii) The system $\{\mathrm{wal}_{\mathbf{k}} : \mathbf{k}\in\mathbb{N}_0^s\}$ is a complete orthonormal system in $L_2([0,1)^s)$.

(cf. [16, Proposition 2.4])

Proof. Item (i) follows immediately from the one-dimensional case (Theorem 2.5) and the definition of multivariate Walsh functions. Item (ii) can easily be derived from Lemma 2.7 by a straightforward application of Fubini and for item (iii) we once again refer to [15, Satz 3]. □

Now we dispose of all necessary requirements to define and work with the weighted Hilbert space $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$.

2.4 The weighted Hilbert space of generalized Walsh series

In this section we will follow the approach used by J. Dick and F. Pillichshammer in [6, Section 2.2] or by G. Pirsic and F. Pillichshammer in [17, pp. 411f.]. This means we introduce the one-dimensional weighted Hilbert space of functions $\mathscr{H}_{\mathrm{wal},\beta,\gamma}$ first and consider the general case later, as many results do not need a lot of improvements in order to be generalized. In fact, the s-dimensional space will simply be defined as the tensor product of s one-dimensional spaces.

2.4.1 The one-dimensional case

From now on let $\beta > 1$. Then, for $\gamma > 0$ and $k \in \mathbb{N}_0$, we define

$$r(\beta, \gamma, k) := \begin{cases} 1 & \text{if } k = 0, \\ \gamma q^{-\beta \lfloor \log_q k \rfloor} & \text{if } k \in \mathbb{N}, \end{cases} \tag{4}$$

(cf. [17, p. 411]).

Again, the dependency of $r(\beta, \gamma, k)$ on q is neglected in this way of notation, as we consider it fixed.

Furthermore, following the discussion from the paragraph after Theorem 2.9, we will now introduce *generalized Walsh series*.

Definition 2.11 (Generalized Walsh series)**.** A *generalized Walsh series* is a function f which is representable by a series of the form

$$f(x) = \sum_{k=0}^{\infty} \hat{f}(k) \operatorname{wal}_k(x),$$

where $x \in [0, 1)$ and $\hat{f}(k) \in \mathbb{C}$ are the so-called *Walsh-Fourier coefficients.*

(cf. [5, Definition A.14])

Once more we would like to point out, that we will usually simply refer to such as *Walsh series.*

The term *Walsh-Fourier coefficient* seems to be chosen rather inappropriately, as, at a first glance, it is not obviously related to a Fourier coefficient. By a short review of the L_2-case (see (3)), however, one can convince oneself of the contrary.

Besides, note that whenever a Walsh series is uniformly convergent we can apply the theorem of dominated convergence to find that

$$\begin{aligned} \hat{f}(k) &\overset{\text{Thm. 2.8}}{=} \sum_{l=0}^{\infty} \hat{f}(l) \int_0^1 \mathrm{wal}_l(x)\overline{\mathrm{wal}_k(x)}\,\mathrm{d}x \\ &= \int_0^1 \sum_{l=0}^{\infty} \hat{f}(l)\,\mathrm{wal}_l(x)\overline{\mathrm{wal}_k(x)}\,\mathrm{d}x \\ &= \int_0^1 f(x)\overline{\mathrm{wal}_k(x)}\,\mathrm{d}x \qquad (5) \end{aligned}$$

for every $k \in \mathbb{N}_0$ (cf. [5, Remark A.15]). Thus we have an analogous result as in (3).

The next step to arrive at a Hilbert space is to define an inner product. So, for Walsh series f and g with Walsh-Fourier coefficients $\hat{f}(k)$ and $\hat{g}(k)$ respectively, $k \in \mathbb{N}_0$, we set

$$\langle f, g\rangle_{\mathrm{wal},\gamma} := \sum_{k=0}^{\infty} r(\beta,\gamma,k)^{-1}\ \hat{f}(k)\ \overline{\hat{g}(k)}.$$

Furthermore, we define

$$\mathscr{H}_{\mathrm{wal},\beta,\gamma} := \left\{ f = \sum_{k=0}^{\infty} \hat{f}(k)\,\mathrm{wal}_k :\ \hat{f}(k) \in \mathbb{C} \text{ and } \langle f, f\rangle_{\mathrm{wal},\gamma} < \infty \right\},$$

(cf. [17, p. 411]).

Indeed, $\langle\cdot,\cdot\rangle_{\mathrm{wal},\gamma}$ is an inner product, as the following lemma shows.

Lemma 2.12. *$\mathscr{H}_{\mathrm{wal},\beta,\gamma}$ being defined as above is a pre-Hilbert space.*

Proof. We need to show that $\langle\cdot,\cdot\rangle_{\mathrm{wal},\gamma}$ satisfies all properties of an inner product. It is obvious that symmetry and linearity in the first argument hold.

For positive definiteness we consider an arbitrary Walsh series

$$f(x) = \sum_{k=0}^{\infty} \hat{f}(k)\,\mathrm{wal}_k(x),$$

$x \in [0,1)$, and, since $r(\beta,\gamma,k) > 0$ for any $k \in \mathbb{N}_0$, we immediately see that

$$\langle f, f\rangle_{\mathrm{wal},\gamma} = \sum_{k=0}^{\infty} r(\beta,\gamma,k)^{-1} \left|\hat{f}(k)\right|^2 \geqslant 0.$$

Now, if we assume $\langle f, f\rangle_{\mathrm{wal},\gamma} = 0$, the above implies that $\hat{f}(k) = 0$ for all $k \in \mathbb{N}_0$. Inserting this into the definition of f shows that f has to be the zero function. □

The following argument is taken from an adendum to [6] by J. Dick.

Remark 2.13. From Lemma 2.12 we deduce that we cannot find two different functions $f, g \in \mathscr{H}_{\mathrm{wal},\beta,\gamma}$ for which $\hat{f}(k) = \hat{g}(k)$ holds for all $k \in \mathbb{N}_0$, where $\hat{f}(k)$, $\hat{g}(k)$ denote the respective Walsh-Fourier coefficients, as this would violate positive definiteness of the inner product. Hence, $\mathscr{H}_{\mathrm{wal},\beta,\gamma}$ contains only those functions f which are equal to $\sum_{k=0}^{\infty} \hat{f}(k)\,\mathrm{wal}_k(x)$ everywhere on $[0,1)$ and which satisfy $\langle f, f\rangle_{\mathrm{wal},\gamma} < \infty$.

The only thing that keeps $\mathscr{H}_{\mathrm{wal},\beta,\gamma}$ from being a Hilbert space is completeness. This, however, is already the case due to the next lemma. As a matter of fact, we will show a more general result which can also be referred to in the higher dimensional case.

Lemma 2.14. *Let $c = (c_k)_{k\in\mathbb{N}_0}$ be a sequence of positive real numbers. On the set*

$$\ell_c^2 := \left\{ x = (x_k)_{k\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}} : \sum_{k=0}^{\infty} c_k |x_k|^2 < \infty \right\}$$

we define an inner product by

$$\langle x, y\rangle := \sum_{k=0}^{\infty} c_k x_k \overline{y_k}.$$

Then ℓ_c^2 is complete w.r.t. $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$.

Proof. (Source: personal communication with G. Leobacher and F. Pillichshammer)

It is clear that $\langle \cdot, \cdot \rangle$ actually is an inner product, as $c_k > 0$ for all $k \in \mathbb{N}_0$, so the definition makes sense.

For the completeness part consider a Cauchy sequence in ℓ_c^2, say $(x^{(n)})_{n\in\mathbb{N}}$. Then, the sequence $(x_k^{(n)})_{n\in\mathbb{N}}$, too, is a Cauchy sequence for any $k \geqslant 0$, as we have

$$c_k \left| x_k^{(n)} - x_k^{(m)} \right|^2 \leqslant \sum_{l=0}^{\infty} c_l \left| x_l^{(n)} - x_l^{(m)} \right|^2 = \left\| x^{(n)} - x^{(m)} \right\|^2$$

for all n, $m \in \mathbb{N}$ and all $k \in \mathbb{N}_0$. Any such sequence $(x_k^{(n)})_{n\in\mathbb{N}}$, however, converges towards a complex number and hence we obtain a sequence, call it $x = (x_k)_{k\in\mathbb{N}_0}$, with

$$x_k = \lim_{n\to\infty} x_k^{(n)}$$

for every $k \in \mathbb{N}_0$.

Now, let $\epsilon > 0$. Since $(x^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence there exists an $n_0 \in \mathbb{N}$ such that for any integers $n, m \geqslant n_0$ we have

$$\sum_{k=0}^{N} c_k \left| x_k^{(n)} - x_k^{(m)} \right|^2 \leqslant \left\| x^{(n)} - x^{(m)} \right\|^2 < \epsilon^2$$

for all $N \in \mathbb{N}_0$. Hence, also

$$\sum_{k=0}^{N} c_k \left| x_k - x_k^{(m)} \right|^2 \leqslant \epsilon^2$$

holds for any non-negative integer N and m sufficiently large. Therefore we obtain

$$\sum_{k=0}^{\infty} c_k \left| x_k - x_k^{(m)} \right|^2 \leqslant \epsilon^2$$

for $m \geqslant n_0$. This implies that $x - x^{(m)} \in \ell_c^2$ and hence $x \in \ell_c^2$. Consequently, $(x^{(n)})_{n\in\mathbb{N}}$ converges (towards x) in ℓ_c^2. $\square$

This allows us to introduce $\mathscr{H}_{\text{wal},\beta,\gamma}$ as a Hilbert space.

Corollary 2.15. *Let $\gamma > 0$ and $\beta > 1$. Then*

$$\mathscr{H}_{\mathrm{wal},\beta,\gamma} := \left\{ f = \sum_{k=0}^{\infty} \hat{f}(k) \,\mathrm{wal}_k : \ \hat{f}(k) \in \mathbb{C} \text{ and } \langle f, f \rangle_{\mathrm{wal},\gamma} < \infty \right\}$$

is a Hilbert space.

(cf. [17, p. 411]).

Proof. From Lemma 2.12 we already know that $\langle \cdot, \cdot \rangle_{\mathrm{wal},\gamma}$ is an inner product on $\mathscr{H}_{\mathrm{wal},\beta,\gamma}$. Furthermore, we can uniquely identify a function $f \in \mathscr{H}_{\mathrm{wal},\beta,\gamma}$ with the sequence of its Walsh-Fourier coefficients, i.e. $\left(\hat{f}(k)\right)_{k \in \mathbb{N}_0}$ (see Remark 2.13), and hence we can apply Lemma 2.14 to find that $\mathscr{H}_{\mathrm{wal},\beta,\gamma}$ is complete. □

Now we consider the function

$$K_{\mathrm{wal},\beta,\gamma}(x,y) := \sum_{k=0}^{\infty} r(\beta,\gamma,k) \,\mathrm{wal}_k(x) \overline{\mathrm{wal}_k(y)},$$

(cf. [17, p. 411]). To verify that this is the reproducing kernel for $\mathscr{H}_{\mathrm{wal},\beta,\gamma}$ we need to prove the following lemma first.

Lemma 2.16. *Let $\gamma > 0$, $\beta > 1$ and $r(\beta,\gamma,k)$ be given as in (4). Then the following identity holds:*

$$\sum_{k=0}^{\infty} r(\beta,\gamma,k) = 1 + \gamma\mu(\beta),$$

where

$$\mu(\beta) := \frac{q^{\beta}(q-1)}{q^{\beta}-q}.$$

(cf. [6, p. 155])

Proof. By the definition of $r(\beta,\gamma,k)$ we have

$$\sum_{k=0}^{\infty} r(\beta,\gamma,k) = 1 + \gamma \sum_{k=1}^{\infty} q^{-\beta \lfloor \log_q k \rfloor}.$$

Now, consider a fixed $k \in \mathbb{N}$. For such a k we can find an $a \in \mathbb{N}_0$ such that $q^a \leqslant k < q^{a+1}$. Therefore it is true that $a \leqslant \log_q(k) < a+1$ and hence $a = \lfloor \log_q k \rfloor$. With this we are able to simplify the above infinite sum in the same way as it was done in [6, p. 155], giving

$$\begin{aligned}\sum_{k=1}^{\infty} q^{-\beta\lfloor \log_q k\rfloor} = \sum_{a=0}^{\infty} q^{-\beta a} \sum_{k=q^a}^{q^{a+1}-1} 1 \quad &= \sum_{a=0}^{\infty} q^{-\beta a}(q^{a+1} - q^a) \\ &= (q-1)\sum_{a=0}^{\infty} \left(q^{1-\beta}\right)^a.\end{aligned}$$

Since $\beta > 1$ we have $q^{1-\beta} < 1$, which means that the above expression is a geometric series. Thus, we finally obtain

$$\sum_{k=0}^{\infty} r(\beta,\gamma,k) = 1 + \gamma(q-1)\frac{1}{1-q^{1-\beta}} = 1 + \gamma\frac{q^{\beta}(q-1)}{q^{\beta}-q} = 1 + \gamma\mu(\beta).$$

□

Theorem 2.17. *The previously defined function*

$$K_{\mathrm{wal},\beta,\gamma}(x,y) = \sum_{k=0}^{\infty} r(\beta,\gamma,k)\,\mathrm{wal}_k(x)\overline{\mathrm{wal}_k(y)}$$

is the reproducing kernel for the weighted Hilbert space $\mathscr{H}_{\mathrm{wal},\beta,\gamma}$.

(cf. [17, p. 411])

Proof.

(RK1) We need to show that for any $y \in [0,1)$ we have $K_{\mathrm{wal},\beta,\gamma}(\cdot,y) \in \mathscr{H}_{\mathrm{wal},\beta,\gamma}$, i.e.

$$\forall y \in [0,1)\colon \quad \|K_{\mathrm{wal},\beta,\gamma}(\cdot,y)\|_{\mathrm{wal},\gamma} < \infty.$$

Obviously, $K_{\mathrm{wal},\beta,\gamma}(\cdot,y)$ is a Walsh series with Walsh-Fourier coefficients

$$\hat{K}_{\mathrm{wal},\beta,\gamma}(\cdot,y)(k) = r(\beta,\gamma,k)\overline{\mathrm{wal}_k(y)} \tag{6}$$

for $y \in [0,1)$ and all $k \in \mathbb{N}_0$. Hence we obtain (cf. [6, p. 155])

$$\begin{aligned}\|K_{\mathrm{wal},\beta,\gamma}(\cdot,y)\|^2_{\mathrm{wal},\gamma} &= \sum_{k=0}^{\infty} r(\beta,\gamma,k)^{-1}\left|\hat{K}_{\mathrm{wal},\beta,\gamma}(\cdot,y)(k)\right|^2 \\ &= \sum_{k=0}^{\infty} r(\beta,\gamma,k)\left|\overline{\mathrm{wal}_k(y)}\right|^2 \\ &= \sum_{k=0}^{\infty} r(\beta,\gamma,k).\end{aligned}$$

As the latter expression equals $1+\gamma\mu(\beta)$ (see Lemma 2.16) it is finite and thus $K_{\mathrm{wal},\beta,\gamma}(\cdot,y) \in \mathscr{H}_{\mathrm{wal},\beta,\gamma}$ for all $y \in [0,1)$.

(RK2) We know that any function $f \in \mathscr{H}_{\mathrm{wal},\beta,\gamma}$ can be written as

$$f(x) = \sum_{k=0}^{\infty} \hat{f}_{\mathrm{wal}}(k)\,\mathrm{wal}_k(x).$$

Therefore, for all $f \in \mathscr{H}_{\mathrm{wal},\beta,\gamma}$ and $y \in [0,1)$ we have

$$\langle f, K_{\mathrm{wal},\beta,\gamma}(\cdot,y)\rangle_{\mathrm{wal},\gamma} \overset{(6)}{=} \sum_{k=0}^{\infty} \hat{f}(k)\,\mathrm{wal}_k(y) = f(y),$$

(cf. [6, p. 155]). This completes the proof.

□

Even though we have found the reproducing kernel for $\mathscr{H}_{\mathrm{wal},\beta,\gamma}$ it is not yet of practical use, as by definition it involves evaluating an infinite sum. Fortunately, $K_{\mathrm{wal},\beta,\gamma}$ possesses the very favorable property that it can be simplified further. So the main emphasis of the subsequent paragraphs will be on attaining a closed form which can be calculated computationally. To this end we follow the steps given in [6, pp. 155f.] and adapt them accordingly.

Using the same trick as in the beginning of the proof of Lemma 2.16 we obtain

$$\begin{aligned}K_{\mathrm{wal},\beta,\gamma}(x,y) &= 1+\gamma\sum_{k=1}^{\infty} q^{-\beta\lfloor\log_q k\rfloor}\,\mathrm{wal}_k(x)\overline{\mathrm{wal}_k(y)} \\ &= 1+\gamma\sum_{a=0}^{\infty}\sum_{k=q^a}^{q^{a+1}-1} q^{-\beta\lfloor\log_q k\rfloor}\,\mathrm{wal}_k(x\ominus y) \\ &= 1+\gamma\sum_{a=0}^{\infty} q^{-\beta a}\sum_{k=q^a}^{q^{a+1}-1}\mathrm{wal}_k(x\ominus y). \qquad (7)\end{aligned}$$

We will now proceed by breaking down the problem, starting with the simplification of

$$D_a(x,y) := \sum_{k=q^a}^{q^{a+1}-1} \mathrm{wal}_k(x \ominus y), \quad a \in \mathbb{N}_0. \tag{8}$$

Lemma 2.18. *Let $a \in \mathbb{N}_0$ and D_a be as in (8). Furthermore, let $0 \leqslant x, y < 1$ with q-adic expansions $x = x_1 q^{-1} + x_2 q^{-2} + \cdots$ and $y = y_1 q^{-1} + y_2 q^{-2} + \cdots$ respectively. Then*

$$D_a(x,y) = \begin{cases} 0 & \exists i \in \{1,\dots,a\}\colon\ x_i \neq y_i, \\ (q-1)q^a & \forall i \in \{1,\dots,a+1\}\colon\ x_i = y_i, \\ -q^a & \text{else.} \end{cases}$$

(cf. [6, p. 156])

Proof. Any $k \in \{q^a, q^a+1, \dots, q^{a+1}-1\}$ has a q-adic expansion of the form $k = \kappa_1 + \kappa_2 q + \ldots + \kappa_a q^{a-1} + \kappa_{a+1} q^a$ with $1 \leqslant \kappa_{a+1} < q$ and $\kappa_i \in \mathbb{Z}_q$ for $1 \leqslant i \leqslant a$. Additionally, we abbreviate $\eta(x_l) - \eta(y_l)$ as $\boldsymbol{\varrho}_l \in \mathbb{Z}_p^r$ for all $1 \leqslant l \leqslant a+1$. Then, similarly to [6, p. 156], we can rewrite $D_a(x,y)$ as

$$\begin{aligned} \sum_{k=q^a}^{q^{a+1}-1} \mathrm{wal}_k(x \ominus y) &= \sum_{k=q^a}^{q^{a+1}-1} \prod_{l=1}^{a+1} \exp\left(\frac{2\pi \mathrm{i}}{p}\, \eta(\kappa_l) \cdot \boldsymbol{\varrho}_l\right) \\ &= \sum_{k=q^a}^{q^{a+1}-1} \exp\left(\frac{2\pi \mathrm{i}}{p}\, \eta(\kappa_1) \cdot \boldsymbol{\varrho}_1\right) \cdots \exp\left(\frac{2\pi \mathrm{i}}{p}\, \eta(\kappa_{a+1}) \cdot \boldsymbol{\varrho}_{a+1}\right) \\ &= \left(\sum_{\kappa_{a+1}=1}^{q-1} \exp\left(\frac{2\pi \mathrm{i}}{p}\, \eta(\kappa_{a+1}) \cdot \boldsymbol{\varrho}_{a+1}\right)\right) \times \\ &\quad \times \prod_{i=1}^{a} \sum_{\kappa_i=0}^{q-1} \exp\left(\frac{2\pi \mathrm{i}}{p}\, \eta(\kappa_i) \cdot \boldsymbol{\varrho}_i\right). \end{aligned}$$

Now, just as it was done in [6, p. 156] as well, we will distinguish between two cases and adapt the respective steps.

Case 1: $\exists i \in \{1,\dots,a\}\colon\ x_i \neq y_i$.

Then, since η is bijective and $\eta(0) = \mathbf{0}$, we have $\boldsymbol{\varrho}_i \in \mathbb{Z}_p^r \setminus \{\mathbf{0}\}$. Or, more precisely, for $\boldsymbol{\varrho}_i = (\varrho_i^{(1)}, \dots, \varrho_i^{(r)})^\top$ we can say that there exists at least one index r_0 such that $\varrho_i^{(r_0)} \neq 0$. W.l.o.g. we set $r_0 = 1$.

Furthermore, observe that the number of elements in $\mathbb{Z}_p^r$ is $p^r = q$. Together with the fact that $\eta\colon \mathbb{Z}_q \to \mathbb{Z}_p^r$ is bijective this means that every element of $\mathbb{Z}_p^r$ appears exactly once in the sum below. Hence, by rearranging of the summands and using the formula for a geometric sum in the last but one step we obtain

$$\begin{aligned}
\sum_{\kappa_i=0}^{q-1} \exp\left(\frac{2\pi \mathrm{i}}{p}\, \eta(\kappa_i)\cdot \boldsymbol{\varrho}_i\right) &= \sum_{\mathbf{z}\in\mathbb{Z}_p^r} \exp\left(\frac{2\pi \mathrm{i}}{p}\, \mathbf{z}\cdot \boldsymbol{\varrho}_i\right) \\
&= \left(\sum_{z_1=0}^{p-1} \exp\left(\frac{2\pi \mathrm{i}}{p}\, z_1 \varrho_i^{(1)}\right)\right) \times \\
&\quad \times \left(\prod_{j=2}^{r} \sum_{z_j=0}^{p-1} \exp\left(\frac{2\pi \mathrm{i}}{p}\, z_j \varrho_i^{(j)}\right)\right) \\
&\overset{\varrho_i^{(1)}\neq 0}{=} \frac{1-\exp\left(2\pi \mathrm{i}\ \varrho_i^{(1)}\right)}{1-\exp\left(\frac{2\pi \mathrm{i}}{p}\ \varrho_i^{(1)}\right)} \times \\
&\quad \times \left(\prod_{j=2}^{r} \sum_{z_j=0}^{p-1} \exp\left(\frac{2\pi \mathrm{i}}{p}\, z_j \varrho_i^{(j)}\right)\right) \\
&= 0.
\end{aligned}$$

Case 2: $\forall i \in \{1,\dots,a\}\colon\ x_i = y_i$.
This means that for every $1 \leqslant i \leqslant a$ we have $\boldsymbol{\varrho}_i = \mathbf{0}$ and therefore

$$D_a(x,y) = q^a \sum_{\kappa_{a+1}=1}^{q-1} \exp\left(\frac{2\pi \mathrm{i}}{p}\, \eta(\kappa_{a+1})\cdot \boldsymbol{\varrho}_{a+1}\right) \tag{9}$$

If also $x_{a+1} = y_{a+1}$, then, clearly, $D_a(x,y) = q^a(q-1)$.

So now we can assume $x_{a+1} \neq y_{a+1}$ and hence $\boldsymbol{\varrho}_{a+1} \neq \mathbf{0}$. By inserting this in (9) we obtain

$$D_a(x,y) = q^a \left(\sum_{\kappa_{a+1}=0}^{q-1} \exp\left(\frac{2\pi \mathrm{i}}{p}\, \eta(\kappa_{a+1})\cdot \boldsymbol{\varrho}_{a+1}\right) - 1\right) = -q^a,$$

as we have already seen in the first case. □

It solely remains to simplify

$$\phi_{\mathrm{wal},\beta}(x,y) := \sum_{a=0}^{\infty} q^{-\beta a} D_a(x,y). \tag{10}$$

Lemma 2.19. *Let $\phi_{\mathrm{wal},\beta}$ be defined by (10) for $x, y \in [0,1)$ with q-adic expansions $x = x_1 q^{-1} + x_2 q^{-2} + \cdots$ and $y = y_1 q^{-1} + y_2 q^{-2} + \cdots$. Then,*

$$\phi_{\mathrm{wal},\beta}(x,y) = \begin{cases} \mu(\beta) & \text{if } x = y, \\ \mu(\beta) - q^{(i_0-1)(1-\beta)}(\mu(\beta)+1) & \text{if } x_{i_0} \neq y_{i_0} \text{ and } \\ & x_i = y_i \text{ for all } i < i_0, \end{cases}$$

where $\mu(\beta)$ is defined as in Lemma 2.16, i.e.

$$\mu(\beta) = \frac{q^{\beta}(q-1)}{q^{\beta}-q}.$$

(cf. [6, p. 156])

Proof. (Taken from [6, p. 156].)

First assume $x = y$. Then, Lemma 2.18 implies that $D_a(x,y) = (q-1)q^a$ and hence

$$\phi_{\mathrm{wal},\beta}(x,y) = (q-1)\sum_{a=0}^{\infty} q^{(1-\beta)a} = \mu(\beta).$$

Let $x = x_1 q^{-1} + x_2 q^{-2} + \cdots$ and $y = y_1 q^{-1} + y_2 q^{-2} + \cdots$ be the q-adic expansions of x and y respectively. If $x \neq y$ then there exists a smallest index i_0 for which $x_{i_0} \neq y_{i_0}$. In this case we obtain $D_a(x,y) = q^a(q-1)$ for $a < i_0 - 1$ and $D_{i_0-1}(x,y) = -q^{i_0-1}$. Furthermore, $D_a(x,y)$ is zero for all $a \geqslant i_0$. Thus, we can rewrite $\phi_{\mathrm{wal},\beta}$ as follows:

$$
\begin{aligned}
\phi_{\mathrm{wal},\beta}(x,y) &= \sum_{a=0}^{\infty} q^{-\beta a} D_a(x,y) \\
&= (q-1)\sum_{a=0}^{i_0-2}(q^{1-\beta})^a - q^{-\beta(i_0-1)}q^{i_0-1} \\
&= (q-1)\frac{1-q^{(1-\beta)(i_0-1)}}{1-q^{1-\beta}} - q^{-\beta(i_0-1)(1-\beta)} \\
&= \frac{(q-1)q^{\beta}}{q^{\beta}-q} - q^{(1-\beta)(i_0-1)}\frac{q^{\beta}(q-1)}{q^{\beta}-q} - q^{(1-\beta)(i_0-1)} \\
&= \mu(\beta) - q^{(1-\beta)(i_0-1)}\left(\mu(\beta)+1\right).
\end{aligned}
$$

□

Putting together the results of the above lemmas we have already found the sought-for closed form of $K_{\mathrm{wal},\beta,\gamma}$. We summarize this fact in the next theorem.

Theorem 2.20. *Let $\beta > 1$. Then*

$$K_{\mathrm{wal},\beta,\gamma}(x,y) = 1 + \gamma\phi_{\mathrm{wal},\beta}(x,y)$$

holds for all x and y in $[0,1)$ with q-adic expansions $x = x_1q^{-1} + x_2q^{-2} + \cdots$ and $y = y_1q^{-1} + y_2q^{-2} + \cdots$, where

$$\phi_{\mathrm{wal},\beta}(x,y) = \begin{cases} \mu(\beta) & \text{if } x = y, \\ \mu(\beta) - q^{(i_0-1)(1-\beta)}(\mu(\beta)+1) & \text{if } x_{i_0} \neq y_{i_0} \text{ and } x_i = y_i \text{ for all } i < i_0 \end{cases}$$

and

$$\mu(\beta) = \frac{q^{\beta}(q-1)}{q^{\beta}-q}.$$

(cf. [6, p. 156])

Proof. Continuing with Equation (7) and inserting the identities from Lemma 2.18 and Lemma 2.19 yields

$$
\begin{aligned}
K_{\mathrm{wal},\beta,\gamma}(x,y) &= 1 + \gamma\sum_{a=0}^{\infty} q^{-\beta a}\sum_{k=q^a}^{q^{a+1}-1} \mathrm{wal}_k(x\ominus y) = 1 + \gamma\sum_{a=0}^{\infty} q^{-\beta a} D_a(x,y) \\
&= 1 + \gamma\phi_{\mathrm{wal},\beta}(x,y)
\end{aligned}
$$

□

Having proven this very useful formula for the reproducing kernel $K_{\mathrm{wal},\beta,\gamma}$ we now move on to the general s-dimensional case.

2.4.2 The s-dimensional case

As it has already been mentioned beforehand, the weighted s-dimensional Hilbert space $\mathscr{H}_{\mathrm{wal},s,\beta,\gamma}$ will simply be defined by the s-fold tensor product of the corresponding one-dimensional spaces. However, it is not clear from the outset that this actually leaves us with a reproducing kernel Hilbert space.

We start off by finding an appropriate inner product. This will be done in a more general way within the next lemma.

Lemma 2.21. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two pre-Hilbert spaces of functions defined on a set X, with the inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ respectively. We denote their tensor product by*

$$\mathscr{H} := \mathscr{H}_1 \otimes \mathscr{H}_2.$$

For

$$f(x_1,x_2) = \sum_{k=1}^{n} f_1^{(k)}(x_1)\; f_2^{(k)}(x_2) \in \mathscr{H}$$

and

$$g(x_1,x_2) = \sum_{l=1}^{m} g_1^{(l)}(x_1)\; g_2^{(l)}(x_2) \in \mathscr{H},$$

where $n,m \in \mathbb{N}$, $x_i \in X$ and $f_i^{(k)}, g_i^{(l)} \in \mathscr{H}_i$ for $1 \leqslant k \leqslant n$, $1 \leqslant l \leqslant m$ and $i = 1,2$, we define

$$\langle f,g\rangle := \sum_{k=1}^{n}\sum_{l=1}^{m}\langle f_1^{(k)}, g_1^{(l)}\rangle_1\;\langle f_2^{(k)}, g_2^{(l)}\rangle_2.$$

Then $\langle\cdot,\cdot\rangle$ is an inner product on $\mathscr{H}$, turning it into a pre-Hilbert space.

(cf. [1, p. 358])

Proof. (Taken from [1, pp. 358f.].)

As f as well as g may admit of various representations of the above kind we need to show that $\langle f, g \rangle$ is invariant under different representations of f and g. For the function f this can be seen by

$$\begin{aligned}
\langle f, g \rangle &= \sum_{k=1}^{n} \sum_{l=1}^{m} \langle f_1^{(k)}, g_1^{(l)} \rangle_1 \; \langle f_2^{(k)}, g_2^{(l)} \rangle_2 \\
&= \sum_{k=1}^{n} \sum_{l=1}^{m} \Big\langle \langle f_1^{(k)} f_2^{(k)}, g_1^{(l)} \rangle_1, g_2^{(l)} \Big\rangle_2 \\
&= \sum_{l=1}^{m} \Big\langle \langle f, g_1^{(l)} \rangle_1, g_2^{(l)} \Big\rangle_2 .
\end{aligned}$$

In an analogous way we can prove that $\langle f, g \rangle$ is well-defined with respect to the representation of g.

Obviously, $\langle \cdot, \cdot \rangle$ is symmetric and linear in the first argument. Thus, it merely remains to show positive definiteness. For this reason we choose an arbitrary representation of $f \in \mathscr{H}$ of the form

$$f(x_1, x_2) = \sum_{k=1}^{n} f_1^{(k)}(x_1) \; f_2^{(k)}(x_2), \text{ where } f_1^{(k)} \in \mathscr{H}_1, \; f_2^{(k)} \in \mathscr{H}_2 \text{ and } x_1, x_2 \in X.$$

As a first step, we orthonormalize the sequences $(f_1^{(k)})_{1 \leqslant k \leqslant n}$ and $(f_2^{(k)})_{1 \leqslant k \leqslant n}$ in the respective spaces and denote the arising sequences by $(\tilde{f}_1^{(k)})_{1 \leqslant k \leqslant n_1}$ and $(\tilde{f}_2^{(k)})_{1 \leqslant k \leqslant n_2}$ respectively. Secondly, we rewrite f as

$$f(x_1, x_2) = \sum_{k=1}^{n_1} \sum_{l=1}^{n_2} a_{k,l} \tilde{f}_1^{(k)}(x_1) \; \tilde{f}_2^{(l)}(x_2)$$

with suitable coefficients $a_{k,l}$. Hence, we may simplify as follows:

$$\begin{aligned}
\langle f, f \rangle &= \left\langle \sum_{k=1}^{n_1} \sum_{l=1}^{n_2} a_{k,l} \tilde{f}_1^{(k)} \tilde{f}_2^{(l)}, \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} a_{i,j} \tilde{f}_1^{(i)} \tilde{f}_2^{(j)} \right\rangle \\
&= \sum_{k=1}^{n_1} \sum_{l=1}^{n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} a_{k,l} \overline{a}_{i,j} \left\langle \tilde{f}_1^{(k)}, \tilde{f}_1^{(i)} \right\rangle_1 \; \left\langle \tilde{f}_2^{(l)}, \tilde{f}_2^{(j)} \right\rangle_2 \\
&= \sum_{k=1}^{n_1} \sum_{l=1}^{n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} a_{k,l} \overline{a}_{i,j} \delta_{k,i} \delta_{l,j} \\
&= \sum_{k=1}^{n_1} \sum_{l=1}^{n_2} |a_{k,l}|^2 \\
&\geqslant 0,
\end{aligned}$$

where δ_{ij} denotes the so-called Kronecker delta, i.e. for $i, j \in \mathbb{N}$ we have

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Furthermore, if we assume that $\langle f, f \rangle = 0$, this immediately implies $a_{k,l} = 0$ for all $1 \leqslant k \leqslant n_1$ and $1 \leqslant l \leqslant n_2$ and therefore $f(x_1, x_2) = 0$ and the result follows. □

In what follows let $\boldsymbol{\gamma} = (\gamma_j)_{1 \leqslant j \leqslant s}$ be a sequence of positive and non-increasing real numbers. Applying the above lemma inductively on the s-fold tensor product of the one-dimensional Hilbert spaces $\mathscr{H}_{\mathrm{wal},\beta,\gamma_j}$, $1 \leqslant j \leqslant s$, i.e.

$$\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}} := \bigotimes_{j=1}^{s} \mathscr{H}_{\mathrm{wal},\beta,\gamma_j} = \mathscr{H}_{\mathrm{wal},\beta,\gamma_1} \otimes \cdots \otimes \mathscr{H}_{\mathrm{wal},\beta,\gamma_s},$$

we get that $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ is a pre-Hilbert space.

For $f = \sum_{k=1}^{n} \prod_{j=1}^{s} f_j^{(k)} \in \mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ and $g = \sum_{l=1}^{m} \prod_{j=1}^{s} g_j^{(l)} \in \mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$, where $f_j^{(k)}, g_j^{(l)} \in \mathscr{H}_{\mathrm{wal},\beta,\gamma_j}$, we can rewrite the inner product resulting from Lemma 2.21, we denote it by $\langle f, g \rangle_{\mathrm{wal},s,\boldsymbol{\gamma}}$, in the following way:

$$\begin{aligned} \langle f, g \rangle_{\mathrm{wal},s,\boldsymbol{\gamma}} &= \sum_{k=1}^{n} \sum_{l=1}^{m} \prod_{j=1}^{s} \langle f_j^{(k)}, g_j^{(l)} \rangle_{\mathrm{wal},\gamma_j} \qquad (11) \\ &= \sum_{k=1}^{n} \sum_{l=1}^{m} \prod_{j=1}^{s} \left(\sum_{i=0}^{\infty} r(\beta, \gamma_j, i)^{-1} \hat{f}_j^{(k)}(i) \overline{\hat{g}_j^{(l)}(i)} \right), \end{aligned}$$

where $\hat{f}_j^{(k)}(i)$ and $\hat{g}_j^{(k)}(i)$ denote the ith Walsh-Fourier coefficient of the functions $f_j^{(k)}$ and $g_j^{(k)}$ respectively.

We use the same notation as in [5, p. 157], namely

$$r(\beta, \boldsymbol{\gamma}, \mathbf{k}) := \prod_{j=1}^{s} r(\beta, \gamma_j, k_j)$$

and

$$\hat{f}(\mathbf{k}) := \prod_{j=1}^{s} \hat{f}_j(k_j)$$

for $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and where $\hat{f}_j(k_j)$ stands for the k_jth Walsh-Fourier coefficient of a function $f_j \in \mathscr{H}_{\mathrm{wal},\beta,\gamma_j}$. This allows us to simplify further and we arrive at the same inner product as given in the aforementioned paper:

$$\begin{aligned}\langle f, g\rangle_{\mathrm{wal},s,\boldsymbol{\gamma}} &= \sum_{k=1}^{n}\sum_{l=1}^{m}\sum_{\mathbf{i}\in\mathbb{N}_0^s} r(\beta,\boldsymbol{\gamma},\mathbf{i})^{-1}\hat{f}^{(k)}(\mathbf{i})\overline{\hat{g}^{(l)}(\mathbf{i})} \\ &= \sum_{\mathbf{i}\in\mathbb{N}_0^s} r(\beta,\boldsymbol{\gamma},\mathbf{i})^{-1}\hat{f}(\mathbf{i})\overline{\hat{g}(\mathbf{i})}.\end{aligned}$$

By the same arguments as we had in Remark 2.13 we can uniquely identify any function $f \in \mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ by its sequence of Walsh-Fourier coefficients $(\hat{f}(\mathbf{k}))_{\mathbf{k}\in\mathbb{N}_0^s}$. Thus, we can exploit Lemma 2.14 again to find that $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ is complete and is hence forming a Hilbert space, which allows us to make the following definition.

Definition 2.22. Let $s \in \mathbb{N}$ and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_s)$ be a sequence of positive and non-increasing real numbers. Then we define the weighted Hilbert space $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ as

$$\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}} := \bigotimes_{j=1}^{s} \mathscr{H}_{\mathrm{wal},\beta,\gamma_j} = \mathscr{H}_{\mathrm{wal},\beta,\gamma_1} \otimes \cdots \otimes \mathscr{H}_{\mathrm{wal},\beta,\gamma_s}.$$

Furthermore, for $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$ we set

$$r(\beta,\boldsymbol{\gamma},\mathbf{k}) := \prod_{j=1}^{s} r(\beta,\gamma_j,k_j)$$

and

$$\hat{f}(\mathbf{k}) := \prod_{j=1}^{s} \hat{f}_j(k_j),$$

where $\hat{f}_j(k_j)$ denotes the k_jth Walsh-Fourier coefficient of $f_j \in \mathscr{H}_{\mathrm{wal},\beta,\gamma_j}$, and define the inner product on $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ by

$$\langle f, g\rangle_{\mathrm{wal},s,\boldsymbol{\gamma}} := \sum_{\mathbf{k}\in\mathbb{N}_0^s} r(\beta,\boldsymbol{\gamma},\mathbf{k})^{-1}\ \hat{f}(\mathbf{k})\overline{\hat{g}(\mathbf{k})}.$$

(Adapted from [17, pp. 411f.].)

The next step is to find a reproducing kernel for $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$.

Theorem 2.23. *The function*

$$K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\mathbf{x},\mathbf{y}) := \prod_{j=1}^{s} K_{\mathrm{wal},\beta,\gamma_j}(x_j,y_j),$$

where $\mathbf{x}$, $\mathbf{y} \in [0,1)^s$, $\mathbf{x} = (x_1,\dots,x_s)$ *and* $\mathbf{y} = (y_1,\dots,y_s)$, *is a reproducing kernel for* $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$.

(cf. [17, p. 411])

Proof. By definition, $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ comprises only functions of the form $f = \sum_{k=1}^{n}\prod_{j=1}^{s} f_j^{(k)}$ with $f_j^{(k)} \in \mathscr{H}_{\mathrm{wal},\beta,\gamma_j}$, $1 \leqslant j \leqslant s$ and $n \in \mathbb{N}$. So, clearly, $K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\cdot,\mathbf{y}) \in \mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ for any $\mathbf{y} \in [0,1)^s$.

Furthermore, for all functions $f \in \mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$, which are certainly of the above type, and for all $\mathbf{y} = (y_1,\dots,y_s)$ in $[0,1)^s$ we have

$$\begin{aligned} f(\mathbf{y}) &= \sum_{k=1}^{n}\prod_{j=1}^{s} f_j^{(k)}(y_j) \\ &= \sum_{k=1}^{n}\prod_{j=1}^{s} \langle f_j^{(k)}, K_{\mathrm{wal},\beta,\gamma_j}(\cdot_j,y_j)\rangle_{\mathrm{wal},\gamma_j} \\ &\overset{(11)}{=} \langle f, K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\cdot,\mathbf{y})\rangle_{\mathrm{wal},s,\boldsymbol{\gamma}}, \end{aligned}$$

since $K_{\mathrm{wal},\beta,\gamma_j}$ is the reproducing kernel of the one-dimensional space $\mathscr{H}_{\mathrm{wal},\beta,\gamma_j}$, $1 \leqslant j \leqslant s$. □

Remark 2.24. The reproducing kernel from the above theorem can be written in several ways, based on Theorem 2.20, as the equations below show (cf. [17, p. 411]):

$$
\begin{aligned}
K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\mathbf{x},\mathbf{y}) &= \prod_{j=1}^{s} K_{\mathrm{wal},\beta,\gamma_j}(x_j,y_j) \\
&= \prod_{j=1}^{s} \left(1+\gamma_j \phi_{\mathrm{wal},\beta}(x_j,y_j)\right) \qquad (12) \\
&= \prod_{j=1}^{s} \sum_{k_j=0}^{\infty} r(\beta,\gamma_j,k_j)\, \mathrm{wal}_{k_j}(x_j)\overline{\mathrm{wal}_{k_j}(y_j)} \\
&= \sum_{\mathbf{k}\in\mathbb{N}_0^s} r(\beta,\boldsymbol{\gamma},\mathbf{k})\, \mathrm{wal}_{\mathbf{k}}(\mathbf{x})\overline{\mathrm{wal}_{\mathbf{k}}(\mathbf{y})}. \qquad (13)
\end{aligned}
$$

So, by (12) we see that $K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$, too, can be computed rather easily (cf. [6, p. 157]) and Equation (13) indicates that we have an analogous form of the reproducing kernel as we had in the one-dimensional case (see Theorem 2.17).

3 Digital (t, m, s)-nets

3.1 Motivation and general construction

In order to proceed towards multivariate integration in the weighted Hilbert space $\mathscr{H}_{\mathrm{wal},s,\beta,\gamma}$ as introduced in Section 2.4, we move on with special types of point sets, namely *digital* (t, m, s)*-nets* over the finite field $\mathbb{F}_q$. For a survey on this topic see [5, Chapter 4.4], for instance. For this choice of point sets we will show error estimations for QMC-rules in the next section. Let us begin by defining the term *elementary interval.*

Definition 3.1 (Elementary interval)**.** Let $s, k \in \mathbb{N}$. Additionally, let $b \geqslant 2$ be an integer. Then we call an interval E an s-dimensional *elementary interval in base* b *of order* k iff there exist non-negative integers $d_1, \ldots, d_s$ and $A_1, \ldots, A_s \in \mathbb{N}_0$ with $\sum_{i=1}^{s} d_i = k$ and $A_i < b^{d_i}$ for all $1 \leqslant i \leqslant s$ such that

$$E = \prod_{i=1}^{s} \left[\frac{A_i}{b^{d_i}}, \frac{A_i + 1}{b^{d_i}}\right).$$

(cf. [5, Definition 3.8])

The notion of introducing (digital) (t, m, s)-nets is to find a finite point set $\mathcal{P}$ which best represents an elementary interval E. By this we mean that the relative number of points in $\mathcal{P} \cap E$ equals the Lebesgue measure of E, i.e. $\lambda(E)$, or, in short:

$$\left|\frac{\#(\mathcal{P} \cap E)}{\#\mathcal{P}} - \lambda(E)\right| = 0, \tag{14}$$

where $\#X$ denotes the cardinality of a finite set X (cf. [5, Definition 4.2]). Noticing that $\lambda(E) = b^{-d_1-\cdots-d_s}$ leads to the following definition of (t, m, s)-nets, which was first given by H. Niederreiter in [13], see also [14].

Definition 3.2 ((t, m, s)-net)**.** Let t, m, s be integers with $s, m \geqslant 1$, $0 \leqslant t \leqslant m$ and $b \geqslant 2$. A point set $\mathcal{P} \subseteq [0, 1)^s$ consisting of exactly b^m points is called a (t, m, s)*-net in base* b iff every elementary interval E of order $m - t$ contains exactly b^t points of $\mathcal{P}$, i.e. $\#(\mathcal{P} \cap E) = b^t$.

(cf. [13, Definition 2.2])

Note that, in this setting, we have

$$\left|\frac{\#(\mathcal{P} \cap E)}{\#\mathcal{P}} - \lambda(E)\right| = \left|\frac{b^t}{b^m} - b^{-(m-t)}\right| = 0,$$

(cf. [13, Remark 2.3]), and thus (t, m, s)-nets fulfill the desired property (14).

For the existence of (t, m, s)-nets for certain choices of the parameters t, m and s as well as for general results on the theory of (t, m, s)-nets see [5, Chapter 4.2] or [13], for example. Here, we only mention two important properties.

Remark 3.3.

- Any point set with b^m elements, $b \geqslant 2$, is at least an (m, m, s)-net in base b (cf. [5, Remark 4.9, item 3]) and
- if the so-called *quality parameter* t of a (t, m, s)-net is small, then it has good distribution properties, as can be seen for example in [5, Chapter 5.5.1].

For practical applications, (t, m, s)-nets in base $b \geqslant 2$ are usually obtained by constructing so-called *digital* (t, m, s)*-nets* (cf. [4, p. 1898]). As our working space is still $\mathscr{H}_{\mathrm{wal},s,\beta,\gamma}$ we restrict ourselves to the case, where $b = q = p^r$, with p prime and $r \in \mathbb{N}$. Also, we need to recall the definition of φ_1, which is a fixed bijection from $\mathbb{Z}_q$ onto $\mathbb{F}_q$ with $\varphi_1(0) = 0$, where 0 denotes the neutral element in the corresponding addiditve group (see the beginning of Section 2.2).

Definition 3.4 (Digital (t, m, s)-net)**.** Let q and φ_1 be defined as above and let s,$m \in \mathbb{N}$. Furthermore, let $C_1, \ldots, C_s$ be $m \times m$ matrices over $\mathbb{F}_q$. For every integer $0 \leqslant h < q^m$ we denote by

$$h = h_1 + h_2 q + \cdots + h_m q^{m-1}$$

its q-adic expansion and identify the vector $\mathbf{h} \in \mathbb{F}_q^m$ with

$$\mathbf{h} = (\varphi_1(h_1), \ldots, \varphi_1(h_m)) .$$

We construct a point set $\mathcal{P}$ consisting of exactly q^m points as follows. For $1 \leqslant j \leqslant s$ we set

1.
$$C_j \mathbf{h}^\top =: \left(y_j^{(1)}(h), \ldots, y_j^{(m)}(h)\right)^\top \in \left(\mathbb{F}_q^m\right)^\top$$

and

2.
$$x_h^{(j)} := \frac{\varphi_1^{-1}(y_j^{(1)}(h))}{q} + \cdots + \frac{\varphi_1^{-1}(y_j^{(m)}(h))}{q^m}.$$

For a fixed $0 \leqslant h < q^m$ we assemble the above quantities into a vector

$$\mathbf{x}_h = \left(x_h^{(1)}, \ldots, x_h^{(s)}\right)$$

and define the point set $\mathcal{P}$ by

$$\mathcal{P} := \{\mathbf{x}_h \colon\ 0 \leqslant h < q^m\}.$$

At this point it should be added that, as we do not put any regularity constraints on the matrices $C_1, \ldots, C_s$, we allow a point to appear more than once in $\mathcal{P}$. So the cardinality of such a point set is always q^m.

Now, if there exists an integer parameter $0 \leqslant t \leqslant m$ such that $\mathcal{P}$ is a (t, m, s)-net in base q, then we call $\mathcal{P}$ a *digital* (t, m, s)*-net* (*over* $\mathbb{F}_q$) with generating matrices C_j, $1 \leqslant j \leqslant s$. Often, we will simply refer to such as *digital nets*, if it is clear or not of importance which parameters t, m and s are taken into consideration.

(cf. [17, Definition 2])

Since the determination of the parameter t is not of concern for this thesis, we refer to [5, Theorem 4.52] for the proof of the following lemma.

Lemma 3.5. *Let $\mathcal{P}$ be a point set constructed according to the above principle, using the generating matrices $C_1, \ldots, C_s \in \mathbb{F}_q^{m \times m}$. Then, $\mathcal{P}$ is a $(m - \varrho, m, s)$-net in base q, where $\varrho = \varrho(C_1, \ldots, C_s)$ is defined as the largest integer for which it holds that for any choice of $d_1, \ldots, d_s \in \mathbb{N}_0$ with $d_1 + \cdots + d_s = \varrho$ we have that*

- *the first d_1 row vectors of C_1 together with*
- *the first d_2 row vectors of C_2 together with*
- ...
- *the first d_s row vectors of C_s*

are linearly independent over $\mathbb{F}_q$.

3.2 The algebraic structure of digital nets

The main advantage of using the concept of digital nets, apart from its good distribution properties (see Remark 3.3 or [5, Chapter 5.5.1], for instance), lies in the fact that, together with the operation $\oplus$ (see Definition 2.4), digital nets form an abelian group, as the lemma below shows.

Theorem 3.6. *Let $\mathcal{P} = \{\mathbf{x}_0, \ldots, \mathbf{x}_{q^m-1}\}$ be a digital (t, m, s)-net over $\mathbb{F}_q$ with generating matrices $C_1, \ldots, C_s$. Then $(\mathcal{P}, \oplus)$ is an abelian group.*

(cf. [5, Lemma 4.72])

Proof. Apparently, due to the definition of $\oplus$, associativity and commutativity hold.

Let $0 \leqslant k, l < q^m$ be integers. Then, using the same notation as in Definition 3.4, the i-th component of the vector $\mathbf{x}_k \in \mathcal{P}$, i.e. $x_k^{(i)}$, is given by

$$x_k^{(i)} = \frac{\varphi_1^{-1}(y_i^{(1)}(k))}{q} + \cdots + \frac{\varphi_1^{-1}(y_i^{(m)}(k))}{q^m}, \qquad 1 \leqslant i \leqslant s.$$

Now, we observe that

$$(\mathbf{x}_k \oplus \mathbf{x}_l)^{(i)} = \sum_{h=1}^{m} a_{i,h} q^{-h},$$

where

$$a_{i,h} = \varphi_1^{-1}\left(\varphi_1 \circ \varphi_1^{-1}(y_i^{(h)}(k)) + \varphi_1 \circ \varphi_1^{-1}(y_i^{(h)}(l))\right) = \varphi_1^{-1}\left(y_i^{(h)}(k) + y_i^{(h)}(l)\right).$$

In the next step we notice that by the definition of digital nets we have

$$C_i(\mathbf{k}+\mathbf{l})^\top = C_i\mathbf{k}^\top + C_i\mathbf{l}^\top = \begin{pmatrix} y_i^{(1)}(k) + y_i^{(1)}(l) \\ \vdots \\ y_i^{(m)}(k) + y_i^{(m)}(l) \end{pmatrix} \in \left(\mathbb{F}_q^m\right)^\top$$

for all $1 \leqslant i \leqslant s$, where $\mathbf{k} = (\varphi_1(k_1), \ldots, \varphi_1(k_m))$ and $\mathbf{l} = (\varphi_1(l_1), \ldots, \varphi_1(l_m))$, $k = k_1 + k_2 q + \cdots k_m q^{m-1}$ and $l = l_1 + l_2 q + \cdots l_m q^{m-1}$. Thus, $(\mathbf{x}_k \oplus \mathbf{x}_l)^{(i)}$ is generated by $C_i(\mathbf{k}+\mathbf{l})^\top$.

Since $\mathbb{F}_q^m$ comprises exactly q^m elements – which at the same time equals the number of points in the digital net – and as we can get any of those elements through $\mathbf{k}+\mathbf{l}$ for suitable $\mathbf{k}, \mathbf{l} \in \mathbb{F}_q^m$, there is a one to one correspondence between $\mathbf{x}_k \oplus \mathbf{x}_l$ and the vectors $C_i(\mathbf{k}+\mathbf{l})^\top$, $1 \leqslant i \leqslant s$. This already proves that $(\mathcal{P}, \oplus)$ is a semi-group.

The neutral element is given by the zero vector $\mathbf{0} \in [0,1)^s$ which is certainly contained in $\mathcal{P}$, since it is obtained by applying the generating matrices to the zero vector in $\mathbb{F}_q^m$. For the inverse of an element $\mathbf{x}_k \in \mathcal{P}$, where $0 \leqslant k < q^m$, we simply need to find the inverse of $\mathbf{k}$ in $\mathbb{F}_q^m$, which is naturally given by $\ominus\mathbf{k}$. Consequently, all group axioms are fulfilled. $\square$

Exploiting the group structure of digital nets over $\mathbb{F}_q$ allows us to prove two very interesting properties.

Theorem 3.7. *Let $\mathcal{P}$ be a digital (t, m, s)-net over $\mathbb{F}_q$ with generating matrices $C_1, \ldots, C_s$. Then the following holds:*

(i) If the points of $\mathcal{P}$ are pairwise different then $(\mathcal{P}, \oplus)$ is isomorphic to $\left(\mathbb{F}_q^m, +\right)$.

(ii) For any $\mathbf{h} \in \mathbb{N}_0^s$ the function $\mathrm{wal}_{\mathbf{h}}$ is a character on $(\mathcal{P}, \oplus)$.

(cf. [5, Lemma 4.72] for (i) and [6, p. 159] for the prime case of (ii))

Proof.

(i) As it was done in the proof of [5, Lemma 4.72] we define the mapping

$$\begin{aligned} \Psi\colon\ \mathbb{F}_q^m &\longrightarrow \mathcal{P} \\ \mathbf{k} &\longmapsto \mathbf{x}_k \end{aligned}$$

where k is given by $k = k_1+k_2q+\cdots+k_mq^{m-1}$ and $\mathbf{k} = (\varphi_1(k_1), \ldots, \varphi_1(k_m))$. From this definition we immediately get that Ψ is well defined.

Moreover, in the proof of Theorem 3.6 we have learned that for any $\mathbf{k}$, $\mathbf{l} \in \mathbb{F}_q^m$ we can generate the point $\mathbf{x}_k \oplus \mathbf{x}_l$ by applying the generating matrices to $\mathbf{k} + \mathbf{l}$. Therefore we have

$$\Psi(\mathbf{k}) \oplus \Psi(\mathbf{l}) = \mathbf{x}_k \oplus \mathbf{x}_l = \Psi(\mathbf{k} + \mathbf{l}).$$

So, Ψ is a homomorphism.

To prove injectivity we assume that $\Psi(\mathbf{k}) = \Psi(\mathbf{l})$, for some $\mathbf{k}$, $\mathbf{l} \in \mathbb{F}_q^m$. This means that $\mathbf{x}_k = \mathbf{x}_l$, where the q-adic digits of k and l are determined by φ_1^{-1} applied to the entries of $\mathbf{k}$ and $\mathbf{l}$ respectively. Since the points of $\mathcal{P}$ are mutually different this implies $k = l$ and consequently $\mathbf{k}$=$\mathbf{l}$.

Additionally, both $\mathbb{F}_q^m$ and $\mathcal{P}$ comprise exactly q^m elements and hence we get that Ψ is bijective.

(ii) Just as in [6, p. 159] we fix $\mathbf{h} \in \mathbb{N}_0^s$. Furthermore, let $\mathcal{P} = \{\mathbf{x}_0, \ldots, \mathbf{x}_{q^m-1}\}$ be a digital net over $\mathbb{F}_q$. Then $\mathrm{wal}_{\mathbf{h}}$ is a character on $(\mathcal{P}, \oplus)$ if and only if

$$\mathrm{wal}_{\mathbf{h}}(\mathbf{x}_k \oplus \mathbf{x}_l) = \mathrm{wal}_{\mathbf{h}}(\mathbf{x}_k)\,\mathrm{wal}_{\mathbf{h}}(\mathbf{x}_l)$$

for any $\mathbf{x}_k$, $\mathbf{x}_l \in \mathcal{P}$. This is true due to Proposition 2.10.(i).

□

At this point, the reader is highly advised to go through the paragraphs preceding Definition 2.3, as the definitions made therein are essential to the proof of the next lemma.

Lemma 3.8. *Let $\mathcal{P} = \{\mathbf{x}_0, \ldots, \mathbf{x}_{q^m-1}\}$ be a digital (t, m, s)-net over $\mathbb{F}_q^m$ generated by the $m \times m$ matrices $C_1, \ldots, C_s \in \mathbb{F}_q^{m\times m}$, $s \in \mathbb{N}$. Additionally, let $\mathbf{k} = (k_1, \ldots, k_s)$ be a vector whose entries are non-negative integers, all of which are strictly smaller than q^m. Then*

$$\sum_{h=0}^{q^m-1} \mathrm{wal}_{\mathbf{k}}(\mathbf{x}_h) = \begin{cases} q^m & \text{if } C_1^\top \varphi(k_1) + \cdots + C_s^\top \varphi(k_s) = \mathbf{0}, \\ 0 & \text{else,} \end{cases}$$

where $\mathbf{0}$ *denotes the zero vector in* $\mathbb{F}_q^m$ *and* $\varphi : \mathbb{Z}_{q^m} \to \left(\mathbb{F}_q^m\right)^\top$ *is an extension of* φ_1 *such that for* $k = \sum_{i=1}^m \kappa_i q^{i-1}$ *we have*

$$\varphi(k) = (\varphi_1(\kappa_1), \ldots, \varphi_1(\kappa_m))^\top .$$

(cf. [16, Lemma 2.5])

Proof. At first, we adopt the preparatory paragraphs from [16, pp. 389f.]. Within this proof we will look at $\mathbb{F}_q$ as a vector space over $\mathbb{Z}_p$. This means we consider $\mathbb{F}_q = \mathbb{Z}_p[\theta]$ so that $B = \{1, \theta, \ldots, \theta^{r-1}\}$ is a basis for $\mathbb{F}_q$. Certainly, any $x \in \mathbb{F}_q$ can then (uniquely) be written as $x = \sum_{i=1}^r x_i \theta^{i-1}$ with $x_i \in \mathbb{Z}_p$ for $i = 1, \ldots, r$.

For such an x we know that the isomorphism $\psi : \mathbb{F}_q \to \mathbb{Z}_p^r$ is given by $\psi(x) = (x_1, \ldots, x_r)$. We now extend ψ to m-dimensional vectors, $m \in \mathbb{N}$, i.e. $\psi : \mathbb{F}_q^m \to \mathbb{Z}_p^{rm}$. Analogous to the original case we define $\eta = \psi \circ \varphi$. In order to provide the reader with a better overview we summarize these relations in a commutative diagramm.

$$\begin{tikzcd}
\mathbb{Z}_{q^m} \arrow[r, "\varphi"] \arrow[dr, "\eta"'] & \mathbb{F}_q^m \arrow[d, "\psi"] \\
& \mathbb{Z}_p^{rm}
\end{tikzcd}$$

Figure 2: Commutative diagram of extensions, (cf. [16, p. 389]).

In the following paragraphs we will define a mapping Ψ of linear transformations over $\mathbb{F}_q$ into the linear transformations over $\mathbb{Z}_p$. For this reason we represent θ^r as a linear combination of the basis elements, i.e.

$$\theta^r = \theta_1 + \theta_2\theta + \cdots + \theta_r\theta^{r-1}, \quad \theta_i \in \mathbb{Z}_p$$

and define the matrix

$$\Theta := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \theta_1 \\ 1 & 0 & 0 & \cdots & 0 & \theta_2 \\ 0 & 1 & 0 & \cdots & 0 & \theta_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & \theta_r \end{pmatrix}.$$

Applying Θ to $\psi(x)$, where x has the basis representation $x = \sum_{i=1}^{r} x_i \theta^{i-1}$, yields

$$\Theta \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} = \begin{pmatrix} \theta_1 x_r \\ x_1 + \theta_2 x_r \\ x_2 + \theta_3 x_r \\ \vdots \\ x_{r-1} + \theta_r x_r \end{pmatrix}.$$

If we now consider the linear transformation $x \mapsto \theta x$ we get

$$\begin{aligned} \theta x &= \sum_{i=1}^{r} x_i \theta^i = \sum_{i=1}^{r-1} x_i \theta^i + x_r \theta^r = \sum_{i=1}^{r-1} x_i \theta^i + x_r(\theta_1 + \theta_2 \theta + \cdots + \theta_r \theta^{r-1}) \\ &= \theta_1 x_r + (x_1 + \theta_2 x_r)\theta + \cdots + (x_{r-1} + \theta_r x_r)\theta^{r-1}. \end{aligned}$$

Comparing the results we have shown the identity

$$\Theta \psi(x) = \psi(\theta x) \tag{15}$$

for all $x \in \mathbb{F}_q$.

For an arbitrary $\alpha \in \mathbb{F}_q$ with the representation $\alpha = \sum_{i=1}^{r} a_i \theta^{i-1}$ with respect to the basis B we define the map Ψ by the matrix

$$\Psi(\alpha) = \sum_{i=1}^{r} a_i \Theta^{i-1}.$$

If we now exploit (15) and the fact that ψ is an isomorphism we can easily show that

$$\begin{aligned} \Psi(\alpha)\psi(x) &= \sum_{i=1}^{r} a_i \Theta^{i-1} \psi(x) = \sum_{i=1}^{r} a_i \psi\left(\theta^{i-1} x\right) = \psi\left(\sum_{i=1}^{r} a_i \theta^{i-1} x\right) \\ &= \psi(\alpha x) \end{aligned} \tag{16}$$

holds for all $x \in \mathbb{F}_q$.

The next step is to extend Ψ to matrices. This can be achieved by applying Ψ to the entries of the respective matrix and subsequently letting the hereby obtained matrices run together. With some abuse of notation, this can be formulated as follows:

$$\Psi(A) := \left(\Psi(a_{i,j})\right)_{i,j} \in \mathbb{Z}_p^{rm_1 \times rm_2} \qquad \text{for } A = \left(a_{i,j}\right)_{i,j} \in \mathbb{F}_q^{m_1 \times m_2},$$

where m_1, $m_2 \in \mathbb{N}$. Again we obtain

$$\Psi(A)\psi(\mathbf{x}) = \psi(A\mathbf{x}), \tag{17}$$

with $\mathbf{x} = (x_1, \ldots, x_{m_2})^\top \in \left(\mathbb{F}_q^{m_2}\right)^\top$, as a consequence of (16) and the homomorphism property of ψ.

We now enter the actual proof of [16, Lemma 2.5]. To this end let $\mathbf{k} = (k_1, \ldots, k_s) \in \{0, \ldots, q^m - 1\}^s$ and let the entry k_j, $1 \leqslant j \leqslant s$, have the q-adic expansion $k_j = \kappa_{j,1} + \kappa_{j,2}q + \cdots + \kappa_{j,m}q^{m-1}$. The ith component of the point $\mathbf{x}_h \in \mathcal{P}$ shall be denoted by $x_h^{(i)}$, $1 \leqslant i \leqslant s$ and $0 \leqslant h < q^m$.

From the construction scheme of digital nets we know that

$$x_h^{(i)} = \frac{\varphi_1^{-1}(y_i^{(1)}(h))}{q} + \cdots + \frac{\varphi_1^{-1}(y_i^{(m)}(h))}{q^m},$$

where $y_i^{(1)}(h), \ldots, y_i^{(m)}(h)$ are as stated in Definition 3.4. Therefore, for any $1 \leqslant l \leqslant m$, the lth q-adic digit of $x_h^{(i)}$, we denote it by $x_{h,l}^{(i)}$, is given by

$$x_{h,l}^{(i)} = \varphi_1^{-1}(y_i^{(l)}(h)) = \varphi_1^{-1}(\mathbf{c}_{i,l}^\top \cdot \varphi(h)),$$

where $\mathbf{c}_{i,l}$ is the lth row vector of the generating matrix C_i. Hence we have

$$\eta(x_{h,l}^{(j)}) = \psi \circ \varphi_1 \circ \varphi_1^{-1}(\mathbf{c}_{j,l}^\top \cdot \varphi(h)) = \psi(\mathbf{c}_{j,l}^\top \cdot \varphi(h)),$$

$1 \leqslant j \leqslant s$. With these presettings we can proceed as follows:

$$\begin{aligned}\sum_{h=0}^{q^m-1} \mathrm{wal}_{\mathbf{k}}(\mathbf{x}_h) &= \sum_{h=0}^{q^m-1} \prod_{j=1}^{s} \prod_{l=1}^{m} \exp\left(\frac{2\pi \mathrm{i}}{p}\, \eta(\kappa_{j,l}) \cdot \eta(x_{h,l}^{(j)})\right) \\ &= \sum_{h=0}^{q^m-1} \prod_{j=1}^{s} \prod_{l=1}^{m} \exp\left(\frac{2\pi \mathrm{i}}{p}\, \eta(\kappa_{j,l}) \cdot \psi\left(\mathbf{c}_{j,l}^\top \cdot \varphi(h)\right)\right).\end{aligned}$$

Since φ is bijective and using the identity given in (17) we obtain further

$$\begin{aligned}\sum_{h=0}^{q^m-1} \mathrm{wal}_{\mathbf{k}}(\mathbf{x}_h) &= \sum_{\mathbf{h} \in \mathbb{F}_q^m} \prod_{j=1}^{s} \prod_{l=1}^{m} \exp\left(\frac{2\pi \mathrm{i}}{p}\, \eta(\kappa_{j,l}) \cdot \psi\left(\mathbf{c}_{j,l}^\top \cdot \mathbf{h}\right)\right) \\ &= \sum_{\mathbf{a} \in \mathbb{Z}_p^{rm}} \prod_{j=1}^{s} \prod_{l=1}^{m} \exp\left(\frac{2\pi \mathrm{i}}{p}\, \eta(\kappa_{j,l}) \cdot \Psi(\mathbf{c}_{j,l})\mathbf{a}\right) \\ &= \sum_{\mathbf{a} \in \mathbb{Z}_p^{rm}} \exp\left(\frac{2\pi \mathrm{i}}{p}\, \mathbf{a} \cdot \left(\sum_{j=1}^{s} \sum_{l=1}^{m} \Psi(\mathbf{c}_{j,l})^\top \eta(\kappa_{j,l})\right)\right).\end{aligned}$$

Now, observe that

$$\sum_{l=1}^{m} \Psi(\mathbf{c}_{j,l})^\top \eta(\kappa_{j,l}) = \left(\Psi(\mathbf{c}_{j,1}^\top), \ldots, \Psi(\mathbf{c}_{j,m}^\top)\right) \left(\eta(\kappa_{j,1}), \ldots, \eta(\kappa_{j,m})\right)^\top = \Psi(C_j^\top)\eta(k_j)$$

and therefore we arrive at

$$\sum_{\mathbf{a}\in\mathbb{Z}_p^{rm}} \exp\left(\frac{2\pi \mathrm{i}}{p}\, \mathbf{a}\cdot\left(\sum_{j=1}^{s} \Psi(C_j^\top)\eta(k_j)\right)\right).$$

Abbreviating the lth component of $\sum_{j=1}^{s} \Psi(C_j^\top)\eta(k_j)$ as ϱ_l and further simplifications yield

$$\begin{aligned}\sum_{h=0}^{q^m-1} \mathrm{wal}_{\mathbf{k}}(\mathbf{x}_h) &= \sum_{(a_1,\ldots,a_{rm})\in\mathbb{Z}_p^{rm}} \exp\left(\frac{2\pi \mathrm{i}}{p}\, a_1\varrho_1\right)\cdots\exp\left(\frac{2\pi \mathrm{i}}{p}\, a_{rm}\varrho_{rm}\right)\\ &= \prod_{l=1}^{rm}\left(\sum_{a=0}^{p-1}\exp\left(\frac{2\pi \mathrm{i}}{p}\, a\varrho_l\right)\right).\end{aligned}$$

In case there exists an index $1 \leqslant l_0 \leqslant rm$ such that $\varrho_{l_0} \neq 0$, then

$$\sum_{a=0}^{p-1}\exp\left(\frac{2\pi \mathrm{i}}{p}\, a\varrho_{l_0}\right) = 0,$$

due to a geometric sum argument and therefore the whole product equals to zero. Clearly, if we have $\varrho_l = 0$ for every $1 \leqslant l \leqslant rm$, we obtain

$$\sum_{h=0}^{q^m-1} \mathrm{wal}_{\mathbf{k}}(\mathbf{x}_h) = \prod_{l=1}^{rm}\left(\sum_{a=0}^{p-1}\exp\left(\frac{2\pi \mathrm{i}}{p}\, a\varrho_l\right)\right) = p^{rm} = q^m.$$

Taking advantage of the fact that ψ is an isomorphism in addition to using (17) and $\eta = \psi\circ\varphi$ the condition $\varrho_l = 0$ holds for all $1 \leqslant l \leqslant rm$ if and only if

$$\mathbf{0} = \sum_{j=1}^{s} \Psi(C_j^\top)\eta(k_j) = \psi\left(\sum_{j=1}^{s} C_j^\top\varphi(k_j)\right),$$

where $\mathbf{0}$ denotes the zero element in $\mathbb{Z}_p^{rm}$.

Note that, if $\psi(x) = \mathbf{0}$, then x has to be zero, as ψ is an isomorphism and therefore the above is true if and only if

$$C_1^\top\varphi(k_1) + \cdots + C_s^\top\varphi(k_s) = \mathbf{0},$$

which proves the statement. □

By having proven this lemma we have finished all preparatory work necessary to deal with the integration problem in the Hilbert space $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$.

4 Multivariate integration in the Hilbert space $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$

We will now focus on the approximation of integrals over the s-dimensional unit cube by applying QMC-rules in the Hilbert space $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$. Apart from proving general bounds for the so-called worst-case error we will also use results of the previous chapters to investigate the application of digital (t,m,s)-nets as sample points. Additionally, some effort will be put into the determination of both necessary and sufficient conditions under which applying a QMC-rule in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ is tractable. That is, whether the number of sample points required to attain a certain error bound is (at most) polynomially dependent on the dimension s and the error itself.

To mathematically formalize the problem we need to introduce two functionals. Let f be a function in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$. Then, by $I_s(f)$ we denote

$$I_s(f) := \int_{[0,1)^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

For the approximation of $I_s(f)$ we use a QMC-rule. This means, we deterministically choose sample points $\mathbf{x}_0,\ldots,\mathbf{x}_{n-1} \in [0,1)^s$ and compute

$$Q_{n,s}(f) := \frac{1}{n}\sum_{h=0}^{n-1} f(\mathbf{x}_h),$$

(cf. [6, p. 161]).

First off, we will prove a very basic, but yet essential property of these functionals.

Lemma 4.1. *Let $\beta > 1$. Then the functionals $I_s(f)$ and $Q_{n,s}(f)$, as defined above, are linear and bounded for $f \in \mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$.*

Proof. Clearly, both I_s and $Q_{n,s}$ are linear. For the proof of boundedness we consider an arbitrary $f(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{N}_0^s} \hat{f}(\mathbf{k})\,\mathrm{wal}_{\mathbf{k}}(\mathbf{x}) \in \mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$. Due to Proposition 2.2.(i) the result follows immediately for $Q_{n,s}$.

Now we consider $I_s(f)$. We have

$$\begin{aligned}|I_s(f)| &\overset{\textbf{(RK2)}}{\leqslant} \int_{[0,1)^s} |\langle f, K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\cdot,\mathbf{y})\rangle_{\mathrm{wal},s,\boldsymbol{\gamma}}| \,\mathrm{d}\mathbf{y} \\ &\leqslant \|f\|_{\mathrm{wal},s,\boldsymbol{\gamma}} \int_{[0,1)^s} \|K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\cdot,\mathbf{y})\|_{\mathrm{wal},s,\boldsymbol{\gamma}} \,\mathrm{d}\mathbf{y}.\end{aligned}$$

From the proof of Theorem 2.17 it then follows that

$$\begin{aligned}\int_{[0,1)^s} \|K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\cdot,\mathbf{y})\|_{\mathrm{wal},s,\boldsymbol{\gamma}} \,\mathrm{d}\mathbf{y} &\leqslant \int_{[0,1)^s} \sqrt{\prod_{j=1}^{s}\left(\sum_{k_j=0}^{\infty} r(\beta,\gamma_j,k_j)\right)}\mathrm{d}\mathbf{y} \\ &\overset{\text{Lemma 2.16}}{=} \sqrt{\prod_{j=1}^{s}(1+\gamma_j\mu(\beta))} < \infty\end{aligned}$$

for $\beta > 1$ and hence we are finished. □

Apart from the fact that we need the boundedness of I_s in order to reasonably consider the integration problem in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$, it allows us to do the following (cf. [5, Example 2.9]):

$$\int_{[0,1)^s} \langle f, K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\cdot,\mathbf{y})\rangle_{\mathrm{wal},s,\boldsymbol{\gamma}} \,\mathrm{d}\mathbf{y} = \left\langle f, \int_{[0,1)^s} K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\cdot,\mathbf{y}) \,\mathrm{d}\mathbf{y}\right\rangle_{\mathrm{wal},s,\boldsymbol{\gamma}}, \tag{18}$$

$f \in \mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$, as follows from the next lemma.

Lemma 4.2. *Let $\mathscr{H}$ be a reproducing kernel Hilbert space of functions with reproducing kernel K and inner product $\langle\cdot,\cdot\rangle$. Furthermore, let T be a linear and bounded functional on $\mathscr{H}$. Then,*

$$T\left(\langle f(x), K(x,y)\rangle_x\right) = \left\langle f(x), \overline{T\left(K(y,x)\right)}\right\rangle_x,$$

where, in the above expression, we make the convention that T is applied to a function in y and that the index of the inner product indicates the variable with respect to which the inner product is taken.

(cf. [5, pp. 25f.])

Proof. (Taken from [5, pp. 25f.])

Since T is linear and bounded by assumption it follows from Riesz' representation theorem that there exists exactly one function $R \in \mathscr{H}$ sucht that for all $f \in \mathscr{H}$ we have that

$$T\left(f(y)\right) = \langle f(y), R(y)\rangle_y.$$

Moreover, for all x in the domain of R

$$R(x) \overset{\textbf{(RK2)}}{=} \langle R(y), K(y,x)\rangle_y = \overline{\langle K(y,x), R(y)\rangle_y} = \overline{T(K(y,x))}$$

holds, as $K(y,x) \in \mathscr{H}$ for any fixed x. Hence,

$$T\left(\langle f(x), K(x,y)\rangle_x\right) \overset{\textbf{(RK2)}}{=} T(f(y)) = \langle f(x), R(x)\rangle_x = \left\langle f(x), \overline{T(K(y,x))}\right\rangle_x.$$

□

The equation given in (18) now follows from the symmetry of reproducing kernels (see Proposition 2.2). By exploiting this result we obtain the following identity:

Lemma 4.3. *The representer of the functional I_s as defined above in the Hilbert space $\mathscr{H}_{\text{wal},s,\beta,\boldsymbol{\gamma}}$ is 1, i.e. for any $f \in \mathscr{H}_{\text{wal},s,\beta,\boldsymbol{\gamma}}$*

$$I_s(f) = \langle f, 1\rangle_{\text{wal},s,\boldsymbol{\gamma}}$$

holds.

(cf. [6, p. 161])

Proof. It is easy to see that the Walsh series of $K_{\text{wal},s,\beta,\boldsymbol{\gamma}}(\mathbf{x},\mathbf{y})$ is uniformly convergent for every fixed $\mathbf{x} \in [0,1)^s$:

$$\left|\sum_{\mathbf{k}\in\mathbb{N}_0^s} r(\beta,\boldsymbol{\gamma},\mathbf{k})\,\text{wal}_{\mathbf{k}}(\mathbf{x})\overline{\text{wal}_{\mathbf{k}}(\mathbf{y})}\right| \overset{\text{Lemma 2.16}}{\leqslant} \prod_{j=1}^{s}\left(1+\gamma_j\mu(\beta)\right) < \infty$$

for $\beta > 1$. Therefore we obtain

$$\int_{[0,1)^s} K_{\text{wal},s,\beta,\boldsymbol{\gamma}}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y} = \sum_{\mathbf{k}\in\mathbb{N}_0^s} r(\beta,\boldsymbol{\gamma},\mathbf{k})\,\text{wal}_{\mathbf{k}}(\mathbf{x})\int_{[0,1)^s}\overline{\text{wal}_{\mathbf{k}}(\mathbf{y})}\,\mathrm{d}\mathbf{y} \overset{\text{Prop. 2.10}}{=} 1,$$

as a consequence of the theorem of dominated convergence. We insert this identity into (18) and together with **(RK2)** the result follows. □

4.1 Error analysis for arbitrary QMC-rules

In the following the definitions of several quality parameters which are of essential concern in this thesis are given. We will now briefly describe two of them. Heuristically speaking, the *worst-case error* gives the largest possible error which can be attained by using a specific QMC-rule for integration, independent of which function in the unit ball of $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ is to be approximated. Whereas *QMC-tractability* is used to examine if there exists a QMC-rule for which one can link the size of the point set necessary to stay below a certain error bound to a polynomial dependency on the dimension, viewed at as a property of $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$.

Definition 4.4 (Worst-case error)**.** Let I_s and $Q_{n,s}$ be as defined in the beginning of this section. The *worst-case error* for integration in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ is defined by

$$e_{n,s} = e(Q_{n,s}) := \sup_{f \in \mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}},\ \|f\|_{\mathrm{wal},s,\boldsymbol{\gamma}} \leqslant 1} |I_s(f) - Q_{n,s}(f)|$$

for $n \in \mathbb{N}$. As a reference value we introduce the *initial error* by

$$e_{0,s} := \sup_{f \in \mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}},\ \|f\|_{\mathrm{wal},s,\boldsymbol{\gamma}} \leqslant 1} |I_s(f)| \,.$$

(cf. [6, Definition 5])

Since, in practice, we can only apply a finite number of sample points to a QMC-algorithm, we are also interested in how many points are necessary to attain a certain error bound.

Definition 4.5 (Information complexity)**.** For $s \in \mathbb{N}$ and real $\epsilon > 0$ we define the *information complexity* $n_{\min}(\epsilon, s)$ by

$$n_{\min}(\epsilon, s) := \min \{ n \in \mathbb{N}_0 \colon \ \exists Q_{n,s} \text{ such that } e(Q_{n,s}) \leqslant \epsilon \ e_{0,s} \} \,.$$

(cf. [6, Definition 5])

From Lemma 4.3 it immediately follows that

$$|I_s(f)| = |\langle f, 1 \rangle_{\mathrm{wal},s,\boldsymbol{\gamma}}| \leqslant \|f\|_{\mathrm{wal},s,\boldsymbol{\gamma}}.$$

Therefore we have

$$e_{0,s} = \sup_{f \in \mathscr{H}_{\mathrm{wal},s,\beta,\gamma},\ \|f\|_{\mathrm{wal},s,\gamma} \leqslant 1} |\langle f, 1 \rangle_{\mathrm{wal},s,\gamma}| = \|1\|^2_{\mathrm{wal},s,\gamma} = 1$$

and hence

$$n_{\min}(\epsilon, s) = \min \{ n \in \mathbb{N}_0 \colon \ \exists Q_{n,s} \text{ such that } e(Q_{n,s}) \leqslant \epsilon \} \, ,$$

where $s \in \mathbb{N}$ and $\epsilon > 0$, (cf. [6, p. 162]). Thus, the information complexity gives the minimal number of function values needed to obtain an ϵ-approximation of an integral with a QMC-algorithm.

Often, one can obtain good asymptotic bounds for the worst-case error. On the basis of these, however, it is not obvious how many sample points are needed to make use of this asymptotic behavior, especially when it comes to QMC-integration in higher dimensions, that is, for large s. This problem is dealt with in the so-called *tractability theory.* Tractability hereby means to have control over the dependency on the dimension and excludes those cases for which $n_{\min}(\epsilon, s)$ grows exponentially in s and ϵ^{-1}.

Definition 4.6 (QMC-tractability)**.** Multivariate integration in $\mathscr{H}_{\mathrm{wal},s,\beta,\gamma}$ is said to be *QMC-tractable* iff

$$\exists a, b, c \in \mathbb{R}, \ a, b, c \geqslant 0 \colon \ \forall s \in \mathbb{N} \ \forall \epsilon \in (0,1) \colon \ n_{\min}(\epsilon, s) \leqslant c s^b \epsilon^{-a}. \tag{19}$$

The infima of a and b such that the above inequality holds are called ϵ- and s-exponents of QMC-tractability.

Furthermore, if (19) holds with $b = 0$, i.e. $n_{\min}$ does not grow by increasing the dimension s, we speak of *strong QMC-tractability.* In this case, the infimum of a is referred to as the ϵ-exponent of strong QMC-tractability.

(cf. [6, Definition 5])

In order to obtain some information on the worst-case error the next step will be to simplify it. For this reason we consider a QMC-rule $Q_{n,s}$ with an arbitrary set of sample points $\mathcal{P} = \{\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}\} \subseteq [0,1)^s$. Exploiting the reproducing property of the reproducing kernel $K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ of $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ and inserting the identity from Lemma 4.3 yields

$$\begin{aligned} I_s(f) - Q_{n,s}(f) &= \langle f, 1\rangle_{\mathrm{wal},s,\boldsymbol{\gamma}} - \frac{1}{n}\sum_{h=0}^{n-1}\langle f, K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\cdot, \mathbf{x}_h)\rangle_{\mathrm{wal},s,\boldsymbol{\gamma}} \\ &= \langle f, 1\rangle_{\mathrm{wal},s,\boldsymbol{\gamma}} - \left\langle f, \frac{1}{n}\sum_{h=0}^{n-1} K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\cdot, \mathbf{x}_h)\right\rangle_{\mathrm{wal},s,\boldsymbol{\gamma}} \\ &= \left\langle f, 1 - \frac{1}{n}\sum_{h=0}^{n-1} K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\cdot, \mathbf{x}_h)\right\rangle_{\mathrm{wal},s,\boldsymbol{\gamma}} \end{aligned}$$

for any $f \in \mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$, (cf. [6, p. 162]).

The main advantage of using the theory of reproducing kernel Hilbert spaces lies in the fact that we can explicitely find a function which is hardest to integrate in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ (cf. [5, p. 28]). This is indicated in the next theorem.

Theorem 4.7. *The worst-case error for integration in the Hilbert space $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ with reproducing kernel $K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ using an arbitrary point set $\mathcal{P} = \{\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}\} \subseteq [0,1)^s$ is given by*

$$e(Q_{n,s}) = \left\| 1 - \frac{1}{n}\sum_{h=0}^{n-1} K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\cdot, \mathbf{x}_h)\right\|_{\mathrm{wal},s,\boldsymbol{\gamma}}.$$

(cf. [6, p. 162])

Proof. (Adapted from [5, p. 28])

We define

$$h(\mathbf{x}) := 1 - \frac{1}{n}\sum_{h=0}^{n-1} K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\mathbf{x}, \mathbf{x}_h)$$

and for $f \in \mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ let

$$R_{n,\mathcal{P}}(f) := |I_s(f) - Q_{n,s}(f)| = \left|\langle f, h\rangle_{\mathrm{wal},s,\boldsymbol{\gamma}}\right|,$$

as we know from the previous paragraph.

First, we consider the case $\|f\|_{\mathrm{wal},s,\boldsymbol{\gamma}} \leqslant 1$. An application of the inequality of Cauchy and Schwarz yields

$$R_{n,\mathcal{P}}(f) = \left|\langle f, h\rangle_{\mathrm{wal},s,\boldsymbol{\gamma}}\right| \leqslant \|f\|_{\mathrm{wal},s,\boldsymbol{\gamma}}\|h\|_{\mathrm{wal},s,\boldsymbol{\gamma}} \leqslant \|h\|_{\mathrm{wal},s,\boldsymbol{\gamma}} \tag{20}$$

as an upper bound for the worst-case error.

For a general $f \in \mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ we therefore get

$$R_{n,\mathcal{P}}\left(\frac{f}{\|f\|_{\mathrm{wal},s,\boldsymbol{\gamma}}}\right) = \frac{\left|\langle f, h\rangle_{\mathrm{wal},s,\boldsymbol{\gamma}}\right|}{\|f\|_{\mathrm{wal},s,\boldsymbol{\gamma}}} \leqslant \frac{\|f\|_{\mathrm{wal},s,\boldsymbol{\gamma}}\|h\|_{\mathrm{wal},s,\boldsymbol{\gamma}}}{\|f\|_{\mathrm{wal},s,\boldsymbol{\gamma}}} = \|h\|_{\mathrm{wal},s,\boldsymbol{\gamma}}. \tag{21}$$

where we have equality for $f = h$. So we can obtain the upper bound from (20) by choosing $f = h$ in (21). Since, of course, $h/\|h\|_{\mathrm{wal},s,\boldsymbol{\gamma}}$ is normed, it is admissible for the supremum used to calculate the worst-case error and hence

$$e(Q_{n,s}) = \|h\|_{\mathrm{wal},s,\boldsymbol{\gamma}}$$

□

Remark 4.8. Based on the result of Theorem 4.7 we can follow the steps given in [6, p. 162] to simplify further, using identities from Remark 2.24 as well as the reproducing property of $K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$:

$$\begin{aligned}
e^2(Q_{n,s}) &= \left\langle 1 - \frac{1}{n}\sum_{h=0}^{n-1} K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\cdot, \mathbf{x}_h), 1 - \frac{1}{n}\sum_{h=0}^{n-1} K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\cdot, \mathbf{x}_h)\right\rangle_{\mathrm{wal},s,\boldsymbol{\gamma}} \\
&= -1 + \frac{1}{n^2}\sum_{h,i=0}^{n-1} K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\mathbf{x}_h, \mathbf{x}_i) \qquad (22) \\
&= -1 + \frac{1}{n^2}\sum_{h,i=0}^{n-1}\prod_{j=1}^{s}\left(1 + \gamma_j \phi_{\mathrm{wal},\beta}(x_h^{(j)}, x_i^{(j)})\right). \qquad (23)
\end{aligned}$$

From (23) we see that the worst-case error in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ using sample points $\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}$ can be calculated with a computational cost of $\mathcal{O}(n^2 s)$ and, as we will see later, this cost can even be reduced to $\mathcal{O}(ns)$ operations if we use digital (t, m, s)-nets (cf. [6, p. 163 and Remark 3]).

By definition, the worst-case error in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ is entirely determined by the point set to which the QMC-rule is applied. Thus, it is legitimate to use the following notation for the worst-case error of a QMC-algorithm with sample points $\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}$:

$$e(Q_{n,s}) = e_{n,s}(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}).$$

This allows us to introduce $\tilde{e}_{n,s}$, the root mean square of the worst-case error, through

$$\tilde{e}_{n,s}^2 := \int_{[0,1)^{ns}} e_{n,s}^2(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1})\, \mathrm{d}\mathbf{x}_0 \ldots \mathrm{d}\mathbf{x}_{n-1},$$

as given in [6, p. 163].

For this entity we are able to prove the following estimate:

Theorem 4.9. *For $\beta > 1$ the root mean square of the worst-case error in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$, as defined above, is bounded by*

$$\tilde{e}_{n,s} \leqslant \frac{1}{n^{1/2}} \exp\left(\frac{\mu(\beta)}{2} \sum_{h=1}^{s} \gamma_j\right),$$

where $\mu(\beta)$ is given by (see Lemma 2.16)

$$\mu(\beta) = \frac{q^\beta(q-1)}{q^\beta - q}.$$

(cf. [6, Theorem 1])

Proof. (Taken from [6, Theorem 1])

Using Equation (22) from Remark 4.8 we can rewrite $\tilde{e}_{n,s}^2$ as follows:

$$\begin{aligned}\tilde{e}_{n,s}^2 &= -1 + \frac{1}{n^2} \sum_{h,i=0}^{n-1} \int_{[0,1)^{2s}} K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\mathbf{x}_h, \mathbf{x}_i)\, \mathrm{d}\mathbf{x}_h\, \mathrm{d}\mathbf{x}_i \\ &= -1 + \frac{1}{n^2}\left(\sum_{h=0}^{n-1} \int_{[0,1)^s} K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\mathbf{x}_h, \mathbf{x}_h)\, \mathrm{d}\mathbf{x}_h \right.\\ &\quad \left. + \sum_{\substack{h,i=0\\ h\neq i}}^{n-1} \int_{[0,1)^{2s}} K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\mathbf{x}_h, \mathbf{x}_i)\, \mathrm{d}\mathbf{x}_h\, \mathrm{d}\mathbf{x}_i\right).\end{aligned}$$

The Walsh series of $K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ is uniformly convergent, as we have already seen in the proof of Lemma 2.12. Thus, the first integral equals

$$\begin{aligned}\int_{[0,1)^s} K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\mathbf{x}_h,\mathbf{x}_h)\,\mathrm{d}\mathbf{x}_h &= \sum_{\mathbf{k}\in\mathbb{N}_0^s} r(\beta,\boldsymbol{\gamma},\mathbf{k}) \int_{[0,1)^s} \mathrm{wal}_{\mathbf{k}}(\mathbf{x})\overline{\mathrm{wal}_{\mathbf{k}}(\mathbf{x})}\,\mathrm{d}\mathbf{x} \\ &= \sum_{\mathbf{k}\in\mathbb{N}_0^s} r(\beta,\boldsymbol{\gamma},\mathbf{k}) \int_{[0,1)^s} \mathrm{wal}_{\mathbf{0}}(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= \sum_{\mathbf{k}\in\mathbb{N}_0^s} r(\beta,\boldsymbol{\gamma},\mathbf{k})\end{aligned}$$

due to Proposition 2.10.(i) and (ii). This, again, can be simplified through Lemma 2.16:

$$\sum_{\mathbf{k}\in\mathbb{N}_0^s} r(\beta,\boldsymbol{\gamma},\mathbf{k}) = \prod_{j=1}^{s}\left(\sum_{k=0}^{\infty} r(\beta,\gamma_j,k)\right) = \prod_{j=1}^{s}\left(1+\gamma_j\mu(\beta)\right).$$

We will now draw our attention to the latter integral, for which we get

$$\begin{aligned}\int_{[0,1)^{2s}} K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\mathbf{x}_h,\mathbf{x}_i)\,\mathrm{d}\mathbf{x}_h\,\mathrm{d}\mathbf{x}_i &= \sum_{\mathbf{k}\in\mathbf{N}_0^s} r(\beta,\boldsymbol{\gamma},\mathbf{k}) \int_{[0,1)^{2s}} \mathrm{wal}_{\mathbf{k}}(\mathbf{x})\overline{\mathrm{wal}_{\mathbf{k}}(\mathbf{y})}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\ &= 1,\end{aligned}$$

as a result of Proposition 2.10 and the uniform convergence of the Walsh series of $K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ again.

Consequently, by inserting these results into the original formula, we obtain

$$\begin{aligned}\tilde{e}_{n,s}^2 &= -1+\frac{1}{n^2}\left(\sum_{h=0}^{n-1}\prod_{j=1}^{s}(1+\gamma_j\mu(\beta)) + \sum_{\substack{h,i=0\\h\neq i}}^{n-1} 1\right) \\ &= -1+\frac{1}{n^2}\left(n\prod_{j=1}^{s}(1+\gamma_j\mu(\beta)) + n(n-1)\right) \\ &= \frac{1}{n}\left(\prod_{j=1}^{s}(1+\gamma_j\mu(\beta)) - 1\right) \\ &\leqslant \frac{1}{n}\exp\left(\mu(\beta)\sum_{j=1}^{s}\gamma_j\right),\end{aligned}$$

where we used the inequality $1+x\leqslant\exp(x)$ for each factor in the last step. $\square$

There is an essential conclusion which can be drawn from this result.

Corollary 4.10. *If $\sum_{j=1}^{\infty} \gamma_j < \infty$, then we have strong QMC-tractability for multivariate integration in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ with an ϵ-exponent of at most 2.*

(cf. [6, Remark 2])

Proof. From Theorem 4.9 we know that

$$\tilde{e}_{n,s}^2 \leqslant \frac{1}{n} \exp\left(\mu(\beta) \sum_{j=1}^{s} \gamma_j\right).$$

As $\tilde{e}_{n,s}^2$ is defined as the mean of $e_{n,s}^2$ over all possible sample points we can deduce that for all $n \in \mathbb{N}$ there exists a point set $\{\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}\} \subseteq [0,1)^s$ such that

$$e_{n,s}^2(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}) \leqslant \frac{1}{n} \exp\left(\mu(\beta) \sum_{j=1}^{s} \gamma_j\right),$$

(cf. [6, Remark 2]).

For an arbitrary $\epsilon \in (0,1)$ let $n_0 \in \mathbb{N}$ such that

$$n_0 < 2 \exp\left(\mu(\beta) \sum_{j=1}^{s} \gamma_j\right) \epsilon^{-2} \leqslant 2n_0.$$

Therefore we have

$$e_{n_0,s}^2(\mathbf{x}_0, \ldots, \mathbf{x}_{n_0-1}) \leqslant \frac{1}{n_0} \exp\left(\mu(\beta) \sum_{j=1}^{s} \gamma_j\right) \leqslant \epsilon^2$$

and hence

$$n_{\min}(\epsilon, s) \leqslant n_0 < 2 \exp\left(\mu(\beta) \sum_{j=1}^{s} \gamma_j\right) \epsilon^{-2}$$

Since we assumed that $\sum_{j=1}^{\infty} \gamma_j < \infty$ and as we know that $\mu(\beta) > 0$ and that $(\gamma_j)_{j \in \mathbb{N}}$ is a sequence of non-negative numbers, we can estimate as follows

$$n_{\min}(\epsilon, s) \leqslant 2 \exp\left(\mu(\beta) \sum_{j=1}^{s} \gamma_j\right) \epsilon^{-2} \leqslant 2 \exp\left(\mu(\beta) \sum_{j=1}^{\infty} \gamma_j\right) \epsilon^{-2} =: c_{\beta,\boldsymbol{\gamma}} \epsilon^{-2},$$

which eliminates the dependency on s. ∎

4.2 Error analysis for digital (t, m, s)-nets over $\mathbb{F}_q$

In this section we investigate which impact the use of digital nets has on the worst-case error and on (strong) tractability respectively. From the definition of digital (t, m, s)-nets over $\mathbb{F}_q$ we know, that the $n = q^m$ sample points are determined by the choice of generating matrices $C_1, \ldots, C_s \in \mathbb{F}_q^{m\times m}$. We will consider this fact by denoting the worst-case error by $e_{q^m,s}(C_1, \ldots, C_s)$ if it stems from a digital (t, m, s)-net over $\mathbb{F}_q$ with generating matrices $C_1, \ldots, C_s$.

At this point, we need to briefly recapitulate some definitions that were made in the course of this thesis. First of all, φ_1 was defined as an arbitrary bijection from $\mathbb{Z}_q$ onto $\mathbb{F}_q$ with $\varphi_1(0) = 0$, where 0 denotes the zero-element of the respective additive group. Additionally, we considered its extension $\varphi\colon \mathbb{Z}_{q^m} \to \mathbb{F}_q^m$ such that for $k \in \mathbb{Z}_{q^m}$ with q-adic expansion $k = \sum_{i=1}^m \kappa_i q^{i-1}$ we have $\varphi(k) = (\varphi_1(\kappa_1), \ldots, \varphi_1(\kappa_m))^\top$.

Moreover, we want to recall, that we identify $\mathbb{Z}_{q^m}$ with the least residue system modulo q^m. Hence, it will not do any harm if we extend φ to non-negative integers by setting $\varphi(k) = \varphi(k \bmod q^m)$, $k \in \mathbb{N}_0$.

Now, we can make the following definition:

Definition 4.11 (Dual net)**.** Let $C_1, \ldots, C_s$ be $m\times m$ matrices over $\mathbb{F}_q$, $s \in \mathbb{N}$. Then, the *dual net* is defined by

$$\mathcal{D} := \left\{\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s\colon\ C_1^\top\varphi(k_1) + \cdots + C_s^\top\varphi(k_s) = \mathbf{0}\right\}.$$

Furthermore, we define

$$\mathcal{D}^* := \mathcal{D}\backslash\{\mathbf{0}\}.$$

(cf. [17, p. 412])

This setting allows us to even simplify the closed form of the worst-case error which we had in Remark 4.8 and reduce its computational cost from $\mathcal{O}(n^2 s)$ to $\mathcal{O}(q^m s) = \mathcal{O}(ns)$, (cf. [6, p. 163 and Remark 3]).

Theorem 4.12. *Let $\mathcal{P} = \{\mathbf{x}_0, \ldots, \mathbf{x}_{q^m-1}\}$ be a digital (t, m, s)-net over $\mathbb{F}_q$ with generating matrices $C_1, \ldots, C_s$. Furthermore, we denote the jth component of the point $\mathbf{x}_h \in \mathcal{P}$, $0 \leqslant h \leqslant q^m - 1$, by $x_h^{(j)}$, for any $1 \leqslant j \leqslant s$. Then we have*

$$e^2_{q^m,s}(C_1, \ldots, C_s) = \sum_{\mathbf{k} \in \mathcal{D}^*} r(\beta, \boldsymbol{\gamma}, \mathbf{k}) = -1 + \frac{1}{q^m} \sum_{h=0}^{q^m-1} \prod_{j=1}^{s} (1 + \gamma_j \phi_{\mathrm{wal},\beta}(x_h^{(j)}, 0)),$$

where $\phi_{\mathrm{wal},\beta}$ is defined as (see Theorem 2.20)

$$\phi_{\mathrm{wal},\beta}(x, y) = \begin{cases} \mu(\beta) & \text{if } x = y, \\ \mu(\beta) - q^{(i_0-1)(1-\beta)}(\mu(\beta) + 1) & \text{if } x_{i_0} \neq y_{i_0} \text{ and } \\ & x_i = y_i \text{ for all } i < i_0, \end{cases}$$

with x_i and y_i denoting the ith digit in the q-adic expansion of x and y respectively.

(cf. [17, p. 412])

Proof. For the first equality we adhere to the proof of [6, Theorem 2]. We start off by inserting the corresponding values into the closed form of the worst-case error which we had in the general case, i.e. Equation (22) in Remark 4.8:

$$\begin{aligned} e^2_{q^m,s}(C_1, \ldots, C_s) &= -1 + \frac{1}{q^{2m}} \sum_{h,i=0}^{q^m-1} K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\mathbf{x}_h, \mathbf{x}_i) \\ &\overset{\text{Rem. 2.24}}{=} -1 + \frac{1}{q^{2m}} \sum_{h,i=0}^{q^m-1} \sum_{\mathbf{k} \in \mathbb{N}_0^s} r(\beta, \boldsymbol{\gamma}, \mathbf{k}) \, \mathrm{wal}_{\mathbf{k}}(\mathbf{x}_h) \overline{\mathrm{wal}_{\mathbf{k}}(\mathbf{x}_i)} \\ &\overset{\text{Prop. 2.10}}{=} -1 + \frac{1}{q^{2m}} \sum_{h,i=0}^{q^m-1} \sum_{\mathbf{k} \in \mathbb{N}_0^s} r(\beta, \boldsymbol{\gamma}, \mathbf{k}) \, \mathrm{wal}_{\mathbf{k}}(\mathbf{x}_h \ominus \mathbf{x}_i). \end{aligned}$$

For a fixed i, $0 \leqslant i \leqslant q^m - 1$, we notice that the mapping $\mathbf{x} \mapsto \mathbf{x} \ominus \mathbf{x}_i$ for $\mathbf{x} \in \mathcal{P}$ is nothing but a permutation of the elements of $\mathcal{P}$, as $(\mathcal{P}, \oplus)$ is a group (see Theorem 3.6). Thus, each summand with respect to i in the above equation yields the same value. Therefore we have

$$e^2_{q^m,s}(C_1,\ldots,C_s) = -1 + \frac{1}{q^m}\sum_{h=0}^{q^m-1}\sum_{\mathbf{k}\in\mathbb{N}_0^s} r(\beta,\boldsymbol{\gamma},\mathbf{k})\,\mathrm{wal}_{\mathbf{k}}(\mathbf{x}_h) \tag{24}$$
$$= -1 + \sum_{\mathbf{k}\in\mathbb{N}_0^s} r(\beta,\boldsymbol{\gamma},\mathbf{k})\,\frac{1}{q^m}\sum_{h=0}^{q^m-1}\mathrm{wal}_{\mathbf{k}}(\mathbf{x}_h).$$

From Lemma 3.8 we know that for $\mathbf{k}=(k_1,\ldots,k_s)$ the sum over h can be reduced to

$$\sum_{h=0}^{q^m-1}\mathrm{wal}_{\mathbf{k}}(\mathbf{x}_h) = \begin{cases} q^m & \text{if } C_1^\top\varphi(k_1)+\cdots+C_s^\top\varphi(k_s)=\mathbf{0},\\ 0 & \text{else.}\end{cases}$$

Together with the fact that $r(\beta,\boldsymbol{\gamma},\mathbf{0})=1$ we arrive at

$$e^2_{q^m,s}(C_1,\ldots,C_s) = \sum_{\mathbf{k}\in\mathcal{D}*} r(\beta,\boldsymbol{\gamma},\mathbf{k}),$$

which proves the first equality.

For the second part we enter the above proof at (24) and exploit the fact that $\mathrm{wal}_{\mathbf{k}}(\mathbf{0})=1$. This gives

$$\begin{aligned} e^2_{q^m,s}(C_1,\ldots,C_s) &= -1 + \frac{1}{q^m}\sum_{h=0}^{q^m-1}\sum_{\mathbf{k}\in\mathbb{N}_0^s} r(\beta,\boldsymbol{\gamma},\mathbf{k})\,\mathrm{wal}_{\mathbf{k}}(\mathbf{x}_h)\overline{\mathrm{wal}_{\mathbf{k}}(\mathbf{0})}\\ &\overset{\text{Rem. 2.24}}{=} -1 + \frac{1}{q^m}\sum_{h=0}^{q^m-1}\prod_{j=1}^{s}(1+\gamma_j\phi_{\mathrm{wal},\beta}(x_h^{(j)},0)).\end{aligned}$$

$\square$

Similarly to Theorem 4.9, we are interested in how the worst-case error behaves in average when using digital nets over $\mathbb{F}_q$ as sample points. This means, since the points of a digital net are already determined by its generating matrices, that we will now consider the mean of the square worst-case error over all possible choices of generating matrices, as the next definition indicates.

Definition 4.13. We denote the set of all possible choices of s $m \times m$ generating matrices over $\mathbb{F}_q$ by

$$\mathcal{C}_q := \left\{(C_1, \ldots, C_s) \colon C_j \in \mathbb{F}_q^{m \times m} \text{ for } 1 \leqslant j \leqslant s\right\}$$

and we define $A_{q^m,s}$ as the mean square worst-case error over $\mathcal{C}_q$, i.e.

$$A_{q^m,s} := \frac{1}{q^{m^2 s}} \sum_{(C_1,\ldots,C_s) \in \mathcal{C}_q} e^2_{q^m,s}(C_1, \ldots, C_s).$$

(cf. [6, pp. 165f.])

Lemma 4.14. *Let $\mu(\beta)$ be defined as in Lemma 2.16 for some $\beta > 1$. Then the following assertions hold true for $A_{q^m,s}$:*

(i)

$$A_{q^m,s} = -1 + \frac{1}{q^m} \prod_{j=1}^{s} (1 + \gamma_j \mu(\beta)) + \left(1 - \frac{1}{q^m}\right) \prod_{j=1}^{s} \left(1 + \gamma_j \frac{\mu(\beta)}{q^{m\beta}}\right)$$

and

(ii)

$$A_{q^m,s} \leqslant \frac{2}{q^m} \prod_{j=1}^{s} (1 + \gamma_j \mu(\beta)).$$

(cf. [6, Lemma 4])

Proof.

(i) (Taken from [6, Lemma 4]).

From Theorem 4.12 we immediately get

$$A_{q^m,s} = \frac{1}{q^{m^2 s}} \sum_{(C_1,\ldots,C_s) \in \mathcal{C}_q} \sum_{\mathbf{k} \in \mathcal{D}^*} r(\beta, \boldsymbol{\gamma}, \mathbf{k}).$$

We transfer the condition for $\mathbf{k} = (k_1, \ldots, k_s)$ to be in $\mathcal{D}^*$ to the sum over $\mathcal{C}_q$, giving

$$A_{q^m,s} = \frac{1}{q^{m^2 s}} \sum_{\mathbf{k} \in \mathbb{N}_0^s \backslash \{\mathbf{0}\}} r(\beta, \boldsymbol{\gamma}, \mathbf{k}) \sum_{\substack{(C_1,\ldots,C_s) \in \mathcal{C}_q \\ C_1^\top \varphi(k_1) + \cdots + C_s^\top \varphi(k_s) = \mathbf{0}}} 1.$$

Thus, the remaining task is to determine the number of all possible choices of generating matrices $(C_1, \ldots, C_s)$ which satisfy $\sum_{j=1}^s C_j^\top \varphi(k_j) = \mathbf{0}$ for given $(k_1, \ldots, k_s) \in \mathbb{N}_0^s \backslash \{\mathbf{0}\}$. To do so, we need to distinguish between two cases.

Case 1: There exists an $\mathbf{l} \in \mathbb{N}_0^s \backslash \{\mathbf{0}\}$, such that $\mathbf{k} = q^m \mathbf{l}$.

Due to the definition of φ, this means that for each k_j, $1 \leqslant j \leqslant s$, we have $\varphi(k_j) = 0$ and hence $\sum_{j=1}^s C_j^\top \varphi(k_j) = \mathbf{0}$ holds independently of $(C_1, \ldots, C_s)$. Therefore, all of the $q^{m^2 s}$ s-tuples of generating matrices fulfill this property.

Case 2: $\mathbf{k} = \mathbf{k}^* + q^m \mathbf{l}$, with $\mathbf{l} \in \mathbb{N}_0^s$ and $\mathbf{k}^* = (k_1^*, \ldots, k_s^*) \in \mathbb{N}_0^s \backslash \{\mathbf{0}\}$, where $0 \leqslant k_j^* < q^m$ for each $1 \leqslant j \leqslant s$.

Inserting k_j into the bijection φ yields $\varphi(k_j) = \varphi(k_j^*)$, $1 \leqslant j \leqslant s$. Therefore, we need to investigate how many $(C_1, \ldots, C_s) \in \mathcal{C}_q$ fulfill $\sum_{j=1}^s C_j^\top \varphi(k_j^*) = \mathbf{0}$.

To this end, we denote by $\mathbf{c}_{j,i}$ the ith row vector of the matrix C_j and we identify each k_j^* with its q-adic expansion $k_j^* = \kappa_{j,1}^* + \kappa_{j,2}^* q + \cdots + \kappa_{j,m}^* q^{m-1}$, where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant s$. Note that $\mathbf{c}_{j,i}^\top$ then is the ith column vector of $C_j^\top$. Thus, the above condition can be rewritten as

$$\sum_{j=1}^{s} \sum_{i=1}^{m} \mathbf{c}_{j,i}^\top \varphi_1(\kappa_{j,i}^*) = \mathbf{0}. \tag{25}$$

Since we assumed $\mathbf{k}^* \neq \mathbf{0}$, we can deduce that there exists an index $1 \leqslant j_0 \leqslant s$ such that $k_{j_0}^* \neq 0$. Hence we can find i_0, $1 \leqslant i_0 \leqslant m$, such that $\kappa_{j_0,i_0}^* \neq 0$ and therefore $\varphi_1(\kappa_{j_0,i_0}^*) \neq 0$.

This in turn implies that for any choice of row vectors $\mathbf{c}_{1,1}, \ldots, \mathbf{c}_{1,m}, \mathbf{c}_{2,1}, \ldots, \mathbf{c}_{j_0,i_0-1}, \mathbf{c}_{j_0,i_0+1}, \ldots, \mathbf{c}_{s,m}$ we can uniquely solve the condition given in (25) for $\mathbf{c}_{j_0,i_0}$, i.e. we have exactly $q^{m^2 s - m}$ admissible elements in $\mathcal{C}_q$ in the second case.

Summarizing the intermediate results from above and abbreviating $\max\{k_1^*, \ldots, k_s^*\}$ as $\|\mathbf{k}^*\|_\infty$ for $\mathbf{k}^* = (k_1^*, \ldots, k_s^*) \in \mathbb{N}_0^s$ yields

$$\begin{aligned}
A_{q^m,s} &= \frac{1}{q^{m^2 s}} \sum_{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\}} r(\beta, \boldsymbol{\gamma}, \mathbf{k}) \sum_{\substack{(C_1,\ldots,C_s) \in \mathcal{C}_q \\ C_1^\top \varphi(k_1) + \cdots + C_s^\top \varphi(k_s) = \mathbf{0}}} 1 \\
&= \frac{1}{q^{m^2 s}} \sum_{\mathbf{l} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\}} r(\beta, \boldsymbol{\gamma}, q^m \mathbf{l})\ q^{m^2 s} \\
&\quad + \frac{1}{q^{m^2 s}} \sum_{\mathbf{l} \in \mathbb{N}_0^s} \sum_{\substack{\mathbf{k}^* \in \mathbb{N}_0^s \setminus \{\mathbf{0}\} \\ \|\mathbf{k}^*\|_\infty < q^m}} r(\beta, \boldsymbol{\gamma}, \mathbf{k}^* + q^m \mathbf{l})\ q^{m^2 s - m} \\
&= -1 + \sum_{\mathbf{l} \in \mathbb{N}_0^s} r(\beta, \boldsymbol{\gamma}, q^m \mathbf{l}) + \frac{1}{q^m} \sum_{\mathbf{l} \in \mathbb{N}_0^s} \sum_{\substack{\mathbf{k}^* \in \mathbb{N}_0^s \setminus \{\mathbf{0}\} \\ \|\mathbf{k}^*\|_\infty < q^m}} r(\beta, \boldsymbol{\gamma}, \mathbf{k}^* + q^m \mathbf{l}).
\end{aligned}$$

We try to simplify these sums separately, starting with the first one. By definition of $r(\beta, \boldsymbol{\gamma}, \mathbf{k})$, see Definition 2.22 and Equation (4), we have

$$\begin{aligned}
\sum_{\mathbf{l} \in \mathbb{N}_0^s} r(\beta, \boldsymbol{\gamma}, q^m \mathbf{l}) &= \prod_{j=1}^{s} \left(\sum_{l_j=0}^{\infty} r(\beta, \gamma_j, q^m l_j) \right) \\
&= \prod_{j=1}^{s} \left(1 + \sum_{l_j=1}^{\infty} q^{-\beta \lfloor \log_q (q^m l_j) \rfloor} \right) \\
&= \prod_{j=1}^{s} \left(1 + \sum_{l_j=1}^{\infty} q^{-\beta (m + \lfloor \log_q l_j \rfloor)} \right) \\
&\overset{\text{Lemma 2.16}}{=} \prod_{j=1}^{s} \left(1 + \gamma_j \frac{\mu(\beta)}{q^{m\beta}} \right). \qquad (26)
\end{aligned}$$

By using this result we obtain for the second sum

$$\begin{aligned}
&\sum_{\mathbf{l} \in \mathbb{N}_0^s} \sum_{\substack{\mathbf{k}^* \in \mathbb{N}_0^s \setminus \{\mathbf{0}\} \\ \|\mathbf{k}^*\|_\infty < q^m}} r(\beta, \boldsymbol{\gamma}, \mathbf{k}^* + q^m \mathbf{l}) \\
&\quad = \sum_{\mathbf{k} \in \mathbb{N}_0^s} r(\beta, \boldsymbol{\gamma}, \mathbf{k}) - \sum_{\mathbf{l} \in \mathbb{N}_0^s} r(\beta, \boldsymbol{\gamma}, q^m \mathbf{l}) \\
&\quad = \prod_{j=1}^{s} (1 + \gamma_j \mu(\beta)) - \prod_{j=1}^{s} \left(1 + \gamma_j \frac{\mu(\beta)}{q^{m\beta}} \right). \qquad (27)
\end{aligned}$$

So, all in all we have

$$\begin{aligned}
A_{q^m,s} &= -1 + \sum_{\mathbf{l}\in\mathbb{N}_0^s} r(\beta,\boldsymbol{\gamma},q^m\mathbf{l}) + \frac{1}{q^m}\sum_{\mathbf{l}\in\mathbb{N}_0^s}\sum_{\substack{\mathbf{k}^*\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}\\ \|\mathbf{k}^*\|_\infty<q^m}} r(\beta,\boldsymbol{\gamma},\mathbf{k}^*+q^m\mathbf{l})\\
&= -1 + \prod_{j=1}^s\left(1+\gamma_j\frac{\mu(\beta)}{q^{m\beta}}\right)\\
&\quad + \frac{1}{q^m}\left(\prod_{j=1}^s(1+\gamma_j\mu(\beta)) - \prod_{j=1}^s\left(1+\gamma_j\frac{\mu(\beta)}{q^{m\beta}}\right)\right)\\
&= -1 + \frac{1}{q^m}\prod_{j=1}^s(1+\gamma_j\mu(\beta)) + \left(1-\frac{1}{q^m}\right)\prod_{j=1}^s\left(1+\gamma_j\frac{\mu(\beta)}{q^{m\beta}}\right),
\end{aligned}$$

which is exactly what we wanted to show.

(ii) First of all, we will show the inequality given in the second part of the proof of [6, Lemma 4], that is

$$-1+\prod_{j=1}^s\left(1+\gamma_j\frac{\mu(\beta)}{q^{m\beta}}\right) \leqslant \frac{1}{q^m}\prod_{j=1}^s(1+\gamma_j\mu(\beta)).$$

This can be seen as follows:

$$\begin{aligned}
-1+\prod_{j=1}^s\left(1+\gamma_j\frac{\mu(\beta)}{q^{m\beta}}\right) &= \sum_{\substack{\mathrm{u}\subseteq\{1,\ldots,s\}\\ \mathrm{u}\neq\emptyset}}\prod_{i\in\mathrm{u}}\gamma_i\frac{\mu(\beta)}{q^{m\beta}}\\
&\overset{\beta>1}{\leqslant} \frac{1}{q^m}\sum_{\substack{\mathrm{u}\subseteq\{1,\ldots,s\}\\ \mathrm{u}\neq\emptyset}}\prod_{i\in\mathrm{u}}\gamma_i\mu(\beta)\\
&\leqslant \frac{1}{q^m}\prod_{j=1}^s(1+\gamma_j\mu(\beta)). \qquad (28)
\end{aligned}$$

Together with part (i) we obtain

$$\begin{aligned}
A_{q^m,s} &= -1 + \frac{1}{q^m}\prod_{j=1}^s(1+\gamma_j\mu(\beta)) + \left(1-\frac{1}{q^m}\right)\prod_{j=1}^s\left(1+\gamma_j\frac{\mu(\beta)}{q^{m\beta}}\right)\\
&\leqslant \frac{1}{q^m}\prod_{j=1}^s(1+\gamma_j\mu(\beta)) - 1 + \prod_{j=1}^s\left(1+\gamma_j\frac{\mu(\beta)}{q^{m\beta}}\right)\\
&\leqslant \frac{2}{q^m}\prod_{j=1}^s(1+\gamma_j\mu(\beta)).
\end{aligned}$$

□

This lemma tells us, that the root mean square average converges at a rate of $\mathcal{O}(q^{-m/2})$. However, we can still improve on the convergence speed under certain conditions. In other words, we can prove the existence of generating matrices for which the worst-case error converges at a speed of $\mathcal{O}(q^{-\beta m+\delta})$ for any $\delta > 0$, (cf. [6, p. 167]).

In the proof of the following theorem we will use the so-called *Jensen's inequality*. It states that for a sequence of non-negative reals $(a_k)_{k\in\mathbb{N}}$ and for any $\lambda \in (0, 1]$ we have that

$$\left(\sum_{k\in\mathbb{N}} a_k\right)^{\lambda} \leqslant \sum_{k\in\mathbb{N}} a_k^{\lambda}, \tag{29}$$

see [6, p. 169], for instance.

Theorem 4.15. *Let $\beta > 1$ and $\lambda \in (1/\beta, 1]$. Then, there exists a digital (t, m, s)-net over $\mathbb{F}_q$ such that*

$$e^2_{q^m,s} \leqslant c_{s,\boldsymbol{\gamma},\lambda,\beta}\, q^{-\frac{m}{\lambda}},$$

where

$$c_{s,\boldsymbol{\gamma},\lambda,\beta} = 2^{\frac{1}{\lambda}} \prod_{j=1}^{s} \left(1 + \gamma_j^{\lambda} \mu(\beta\lambda)\right)^{\frac{1}{\lambda}} \tag{30}$$

and μ is defined as in Lemma 2.16.

(cf. [6, Theorem 3, item 1])

Proof. (Taken from [6, Theorem 3, item 1]).

Before we start with the proof we need to fiddle with some notational matters first. Although it was not explicitely mentioned beforehand, by definition, the worst-case error depends on β and $\boldsymbol{\gamma}$. We will now indicate this by writing $e_{q^m,s}(\beta, \boldsymbol{\gamma})$. Furthermore, for $(\gamma_1^{\lambda}, \gamma_2^{\lambda}, \ldots)$ we will simply write $\boldsymbol{\gamma}^{\lambda}$.

From Theorem 4.12 we know that for any digital net over $\mathbb{F}_q$ we have

$$\begin{aligned} e^2_{q^m,s}(\beta,\boldsymbol{\gamma}) &= \sum_{\mathbf{k}\in\mathcal{D}^*} r(\beta,\boldsymbol{\gamma},\mathbf{k}) \overset{(29)}{\leqslant} \left(\sum_{\mathbf{k}\in\mathcal{D}^*} r(\beta,\boldsymbol{\gamma},\mathbf{k})^\lambda\right)^{\frac{1}{\lambda}} \\ &= \left(\sum_{(k_1,\ldots,k_s)\in\mathcal{D}^*} \prod_{j=1}^s \gamma_j^\lambda q^{-\beta\lambda\lfloor\log_q k_j\rfloor}\right)^{\frac{1}{\lambda}} \\ &= \left(\sum_{\mathbf{k}\in\mathcal{D}^*} r(\beta\lambda,\boldsymbol{\gamma}^\lambda,\mathbf{k})\right)^{\frac{1}{\lambda}} \\ &= \left(e^2_{q^m,s}(\beta\lambda,\boldsymbol{\gamma}^\lambda)\right)^{\frac{1}{\lambda}}. \end{aligned} \tag{31}$$

It is to mention that the use of Jensen's inequality was justifiable, as $r(\beta,\boldsymbol{\gamma},\mathbf{k}) \geqslant 0$ and $1/\beta < \lambda \leqslant 1$.

Lemma 4.14 implies – note that $A_{q^m,s}$ denotes the mean of the square worst-case error with respect to all possible choices of generating matrices – that there exists a digital (t,m,s)-net over $\mathbb{F}_q$ such that

$$e^2_{q^m,s}(\beta\lambda,\boldsymbol{\gamma}^\lambda) \leqslant \frac{2}{q^m}\prod_{j=1}^s\left(1+\gamma_j^\lambda\mu(\beta\lambda)\right).$$

Since $\beta\lambda > 1$, the last expression makes perfect sense ($\mu(\alpha)$ is only defined for $\alpha > 1$).

Finally, by putting together the results we have shown so far, we obtain

$$e^2_{q^m,s}(\beta,\boldsymbol{\gamma}) \leqslant \left(e^2_{q^m,s}(\beta\lambda,\boldsymbol{\gamma}^\lambda)\right)^{\frac{1}{\lambda}} \leqslant \frac{2^{1/\lambda}}{q^{m/\lambda}}\prod_{j=1}^s\left(1+\gamma_j^\lambda\mu(\beta\lambda)\right)^{\frac{1}{\lambda}} = c_{s,\boldsymbol{\gamma},\lambda,\beta}\, q^{-\frac{m}{\lambda}}.$$

□

This allows us to derive certain conditions under which integration in the Hilbert space $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ is (strongly) tractable, as will be shown in the following two corollaries.

Corollary 4.16. *Under the assumption that for some* $\lambda \in (1/\beta, 1]$

$$\sum_{j=1}^{\infty} \gamma_j^{\lambda} < \infty$$

holds, there exists a digital (t, m, s)*-net over* $\mathbb{F}_q$ *such that*

$$e^2_{q^m,s} \leqslant c_{\infty,\boldsymbol{\gamma},\lambda,\beta}\, q^{-\frac{m}{\lambda}} < \infty,$$

where $c_{\infty,\boldsymbol{\gamma},\lambda,\beta}$ *is (formally) given by (30). Hence, integration in* $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ *is strongly QMC-tractable.*

Furthermore, if

$$\lambda_0 := \inf \left\{ \lambda \in (1/\beta, 1] \colon \; \sum_{j=1}^{\infty} \gamma_j^{\lambda} < \infty \right\},$$

then the ϵ*-exponent of strong QMC-tractability is at most* $2\lambda_0$*.*

(cf. [6, Theorem 3, item 2])

Proof. At first we will show that

$$c_{\infty,\boldsymbol{\gamma},\lambda,\beta} < \infty.$$

To this end, we follow the proof of [6, Theorem 3, item 2]. Assuming that $\lambda \in (1/\beta, 1]$ we obtain

$$\begin{aligned} c_{\infty,\boldsymbol{\gamma},\lambda,\beta} &\overset{(30)}{=} 2^{\frac{1}{\lambda}} \prod_{j=1}^{\infty} \left(1 + \gamma_j^{\lambda} \mu(\beta\lambda)\right)^{\frac{1}{\lambda}} \\ &= 2^{\frac{1}{\lambda}} \exp\left(\frac{1}{\lambda} \sum_{j=1}^{\infty} \log\left(1 + \gamma_j^{\lambda} \mu(\beta\lambda)\right) \right). \end{aligned}$$

Since $\sum_{j=1}^{\infty} \gamma_j^{\lambda} < \infty$ and for all $x > -1$ it is true that $\log(1 + x) \leqslant x$, we obtain

$$c_{\infty,\boldsymbol{\gamma},\lambda,\beta} \leqslant 2^{\frac{1}{\lambda}} \exp\left(\frac{\mu(\beta\lambda)}{\lambda} \sum_{j=1}^{\infty} \gamma_j^{\lambda} \right) < \infty.$$

Trivially, we have $1+\gamma_j^\lambda \mu(\beta\lambda) \geqslant 1$ for $\gamma_j > 0$ (the latter is required by definition of $\mathscr{H}_{\text{wal},s,\beta,\boldsymbol{\gamma}}$) and therefore $c_{s,\boldsymbol{\gamma},\lambda,\beta} \leqslant c_{\infty,\boldsymbol{\gamma},\lambda,\beta}$ for any $s \in \mathbb{N}$. Together with Theorem 4.15 this implies

$$e^2_{q^m,s} \leqslant c_{s,\boldsymbol{\gamma},\lambda,\beta}\; q^{-\frac{m}{\lambda}} \leqslant c_{\infty,\boldsymbol{\gamma},\lambda,\beta}\; q^{-\frac{m}{\lambda}} < \infty,$$

which proves the first assertion.

For the second part let $\epsilon \in (0,1)$ be fixed. For this ϵ choose $m \in \mathbb{N}$ such that

$$q^{m-1} < \underbrace{\left(c_{\infty,\boldsymbol{\gamma},\lambda,\beta}\; \epsilon^{-2}\right)^\lambda}_{=:M} \leqslant q^m.$$

Then we have

$$e_{q^m,s} \leqslant c^{\frac{1}{2}}_{\infty,\boldsymbol{\gamma},\lambda,\beta}\; q^{-\frac{m}{2\lambda}} \leqslant \epsilon$$

and hence

$$n_{\min}(\epsilon,s) \leqslant q^m < qM = q\left(c_{\infty,\boldsymbol{\gamma},\lambda,\beta}\; \epsilon^{-2}\right)^\lambda.$$

Therefore we have found an upper bound for $n_{\min}(\epsilon,s)$ which is independent of s and (at most) polynomially dependent on ϵ, implying strong tractability.

Furthermore, if we denote the infimum of all λ satisfying $\sum_{j=1}^\infty \gamma_j^\lambda < \infty$ by λ_0, it immediately follows that the ϵ-exponent is $2\lambda_0$, at most. □

Remark 4.17. If q is a prime number, then the ϵ-exponent is always at least $2/\beta$. This follows from Theorem 19 in *Discrepancy theory and quasi-Monte Carlo integration* by J. Dick and F. Pillichshammer, which is to appear in *A panorama in discrepancy theory.*

Corollary 4.18. *Assuming*

$$A := \limsup_{s\to\infty} \frac{\sum_{j=1}^s \gamma_j}{\log s} < \infty,$$

there exists a digital (t,m,s)-net over $\mathbb{F}_q$ and a constant c_δ which is solely dependent on an arbitrary $\delta > 0$ such that

$$e^2_{q^m,s} \leqslant c_\delta s^{\mu(\beta)(A+\delta)} q^{-m}.$$

Thus, under this condition, integration in $\mathscr{H}_{\text{wal},s,\beta,\boldsymbol{\gamma}}$ is QMC-tractable with an s-exponent of at most $\mu(\beta)A$ and an ϵ-exponent of at most 2.

(cf. [6, Theorem 3, item 3])

Proof. Again, for the estimation of the worst-case error, we proceed as in the proof of [6, Theorem 3]. Provided that $A = \limsup_{s\to\infty} \frac{\sum_{j=1}^{s} \gamma_j}{\log s} < \infty$, for any $\delta > 0$ we can find an s_δ such that for all $s \geqslant s_\delta$ we have

$$\sum_{j=1}^{s} \gamma_j \leqslant (A + \delta) \log s. \tag{32}$$

Inserting $\lambda = 1$ in (30) in Theorem 4.15 and using $\log(1 + x) \leqslant x$ for all $x > -1$ yields

$$\begin{aligned} c_{s,\boldsymbol{\gamma},1,\beta} &= 2\prod_{j=1}^{s} (1 + \gamma_j \mu(\beta)) \\ &= 2s^{\sum_{j=1}^{s} \frac{\log(1+\gamma_j\mu(\beta))}{\log s}} \\ &\leqslant 2s^{\frac{\mu(\beta)\sum_{j=1}^{s}\gamma_j}{\log s}} \\ &\overset{(32)}{\leqslant} 2s^{\mu(\beta)(A+\delta)} \end{aligned}$$

for s sufficiently large. In other words, there exists a positive constant c_δ such that

$$c_{s,\boldsymbol{\gamma},1,\beta} \leqslant c_\delta s^{\mu(\beta)(A+\delta)}$$

for all $s \in \mathbb{N}$. Considering this inequality in Theorem 4.15 finishes the first part of the proof.

For an arbitrary $\epsilon \in (0, 1)$ we choose $m \in \mathbb{N}$ such that

$$q^{m-1} < c_\delta s^{\mu(\beta)(A+\delta)} \epsilon^{-2} \leqslant q^m$$

and continue as in the proof of Corollary 4.16 to find that

$$n_{\min}(\epsilon, s) \leqslant q c_\delta s^{\mu(\beta)(A+\delta)} \epsilon^{-2}.$$

Therefore, if $A < \infty$, integration in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ is QMC-tractable with an ϵ-exponent of at most two. Furthermore, since the exponents of tractability were defined as infima and as $\delta > 0$ was chosen arbitrarily, the s-exponent is at most $\mu(\beta)A$. □

What we have shown so far is, that there exists at least one digital (t, m, s)-net which fulfills a rather favourable error estimation (cf. Theorem 4.15). Up to this point, however, we do not have any idea how many digital nets can meet a certain error barrier. To describe this problem in a more mathematical way we consider the set of all possible choices of s generating matrices over $\mathbb{F}_q$ (see Definition 4.13)

$$\mathcal{C}_q := \left\{(C_1, \dots, C_s) \colon C_j \in \mathbb{F}_q^{m \times m} \text{ for } 1 \leqslant j \leqslant s\right\}$$

and the equiprobable measure ν on $\mathcal{C}_q$, which is defined by

$$\nu(C_1, \dots, C_s) = \frac{1}{q^{m^2 s}} \qquad \forall\ (C_1, \dots, C_s) \in \mathcal{C}_q.$$

For $c > 1$ and $1/\beta < \lambda \leqslant 1$ we define

$$\mathcal{C}_q(c, \lambda) := \left\{(C_1, \dots, C_s) \in \mathcal{C}_q \colon\ e_{q^m,s}(C_1, \dots, C_s) \leqslant c^{\frac{1}{\lambda}} \sqrt{c_{s,\boldsymbol{\gamma},\lambda,\beta} q^{-\frac{m}{\lambda}}}\right\},$$

i.e. all choices of s generating matrices for which the worst-case error worsens at most by the factor $c^{1/\lambda}$ compared to that in Theorem 4.15. Thus, we are interested in how large $\mathcal{C}_q(c, \lambda)$ actually is with respect to ν (cf. [6, p. 168]).

Theorem 4.19. *Let $c > 1$ and $1/\beta < \lambda \leqslant 1$. Using the definitions from the paragraph above we get*

$$\nu\left(\mathcal{C}_q(c, \lambda)\right) > 1 - c^{-2}.$$

(cf. [6, Theorem 3, item 4])

Proof. (Adapted from [6, Theorem 3, item 4])

The proof of this theorem will be split into two parts:

1. We will define a set $\tilde{\mathcal{C}}_q(c) \subseteq \mathcal{C}_q$, for which $\nu(\tilde{\mathcal{C}}_q(c)) > 1 - c^{-2}$ holds.
2. Subsequently, we will show that $\tilde{\mathcal{C}}_q(c) \subseteq \mathcal{C}_q(c, \lambda)$.

ad 1. For the worst-case error in $\mathscr{H}_{\mathrm{wal},s,\beta,\gamma}$ with $\beta > 1$ and weights $\gamma_j > 0$, $1 \leqslant j \leqslant s$, using a digital net over $\mathbb{F}_q$ with generating matrices $C_1, \dots, C_s$ we write $e_{q^m,s}(\beta, \boldsymbol{\gamma}, (C_1, \dots, C_s))$. Furthermore, we define

$$\begin{aligned}\tilde{\mathcal{C}}_q(c) := \Bigg\{ &(C_1, \dots, C_s) \in \mathcal{C}_q \colon\ e_{q^m,s}\left(\beta\lambda, \boldsymbol{\gamma}^\lambda, (C_1, \dots, C_s)\right) \\ &\leqslant c \frac{\sqrt{2}}{q^{m/2}} \prod_{j=1}^{s} \left(q + \gamma_j^\lambda \mu(\beta\lambda)\right)^{\frac{1}{2}} \Bigg\}.\end{aligned}$$

Now we can estimate the mean square worst-case error over $\mathcal{C}_q$, i.e. $A_{q^m,s}$ (cf. Definition 4.13), with parameters $\beta\lambda$ and $\boldsymbol{\gamma}^\lambda$ as follows:

$$\begin{aligned} A_{q^m,s} &= \frac{1}{q^{m^2 s}} \sum_{(C_1,\ldots,C_s)\in\mathcal{C}_q} e^2_{q^m,s}\left(\beta\lambda, \boldsymbol{\gamma}^\lambda, (C_1,\ldots,C_s)\right) \\ &\geqslant \frac{1}{q^{m^2 s}} \sum_{(C_1,\ldots,C_s)\in\mathcal{C}_q\setminus\tilde{\mathcal{C}}_q(c)} e^2_{q^m,s}\left(\beta\lambda, \boldsymbol{\gamma}^\lambda, (C_1,\ldots,C_s)\right) \\ &> \frac{1}{q^{m^2 s}} \sum_{(C_1,\ldots,C_s)\in\mathcal{C}_q\setminus\tilde{\mathcal{C}}_q(c)} c^2 \frac{2}{q^m} \prod_{j=1}^{s} \left(1+\gamma_j^\lambda \mu(\beta\lambda)\right) \\ &= c^2 \frac{2}{q^m} \prod_{j=1}^{s} \left(1+\gamma_j^\lambda \mu(\beta\lambda)\right) \nu\left(\mathcal{C}_q\setminus\tilde{\mathcal{C}}_q(c)\right). \end{aligned}$$

From Lemma 4.14 we know that

$$A_{q^m,s} \leqslant \frac{2}{q^m} \prod_{j=1}^{s} \left(1+\gamma_j^\lambda \mu(\beta\lambda)\right),$$

if we use the parameters $\beta\lambda$ and $\boldsymbol{\gamma}^\lambda$ instead of β and $\boldsymbol{\gamma}$ respectively. Exploiting this fact and using the identity

$$\nu\left(\mathcal{C}_q\setminus\tilde{\mathcal{C}}_q(c)\right) = 1 - \nu\left(\tilde{\mathcal{C}}_q(c)\right)$$

finally yields

$$\nu\left(\tilde{\mathcal{C}}_q(c)\right) > 1 - c^{-2}.$$

ad 2. For any $(C_1,\ldots,C_s)\in\mathcal{C}_q$ the inequality given in (31) in Theorem 4.15 tells us that

$$e(\beta, \boldsymbol{\gamma}, (C_1,\ldots,C_s)) \leqslant e^{\frac{1}{\lambda}}(\beta\lambda, \boldsymbol{\gamma}^\lambda, (C_1,\ldots,C_s)).$$

Thus, if we assume $(C_1,\ldots,C_s)\in\tilde{\mathcal{C}}_q(c)$ for $c > 1$, we have

$$e^2(\beta, \boldsymbol{\gamma}, (C_1,\ldots,C_s)) \leqslant c^{\frac{2}{\lambda}} q^{-\frac{m}{\lambda}} \prod_{j=1}^{s} \left(1+\gamma_j^\lambda \mu(\beta\lambda)\right)^{\frac{1}{\lambda}} = c^{\frac{2}{\lambda}} c_{s,\boldsymbol{\gamma},\lambda,\beta} q^{-\frac{m}{\lambda}}$$

by definition of $\tilde{\mathcal{C}}_q(c)$ and hence $(C_1,\ldots,C_s)\in\mathcal{C}_q(c,\lambda)$.

By combining the results of 1. and 2. we have shown that

$$\nu\left(\mathcal{C}_q(c,\lambda)\right) \geqslant \nu\left(\tilde{\mathcal{C}}_q(c)\right) > 1 - c^{-2}.$$

□

To illustrate the significance of this result we for instance consider $c = 10$. As $\lambda > 1/\beta$ we can say that for at least 99 % of all choices of s generating matrices the worst-case error increases at most by a factor of $10^{1/\lambda} \leqslant 10^{\beta}$, (cf. [6, Remark 4]).

We want to close this section by giving both necessary and sufficient conditions for (strong) QMC-tractability. Before we do so, however, we try to obtain a lower bound for the worst-case error.

Theorem 4.20. *The worst-case error in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ for a QMC-algorithm using the sample points $\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}$ is bounded from below by*

$$e_{n,s}^2\left(\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}\right) \geqslant -1 + \frac{1}{n}\prod_{j=1}^{s}\left(1 + \min(\gamma_j, 1)\mu(\beta)\right).$$

(cf. [6, Theorem 4])

Proof. (Taken from [6, pp. 170f.]).

We begin by showing that $K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}} \geqslant 0$ if $\gamma_j > 0$ for all $1 \leqslant j \leqslant s$. From Equation (12) in Remark 2.24 we know that the reproducing kernel can be written as

$$K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}(\mathbf{x}, \mathbf{y}) = \prod_{j=1}^{s}\left(1 + \gamma_j \phi_{\mathrm{wal},\beta}(x^{(j)}, y^{(j)})\right),$$

for $\mathbf{x} = (x^{(1)}, \ldots, x^{(s)}) \in [0,1)^s$ and $\mathbf{y} = (y^{(1)}, \ldots, y^{(s)}) \in [0,1)^s$ and where

$$\phi_{\mathrm{wal},\beta}(x, y) = \begin{cases} \mu(\beta) & \text{if } x = y, \\ \mu(\beta) - q^{(i_0-1)(1-\beta)}(\mu(\beta) + 1) & \text{if } x_{i_0} \neq y_{i_0} \text{ and } \\ & x_i = y_i \text{ for all } i < i_0 \end{cases}$$

(see Lemma 2.19). The numbers x_i and y_i hereby stem from the q-adic expansions of x and y respectively.

Since $\beta > 1$ and $q \geqslant 2$ we can estimate as follows:

$$\mu(\beta) \overset{\text{Lemma 2.16}}{=} \frac{q^{\beta}(q-1)}{q^{\beta} - q} \geqslant \frac{q^{\beta}}{q^{\beta} - q} > 1.$$

As $q^{1-\beta} < 1$ and $i_0 \geqslant 1$ we arrive at

$$\begin{aligned}\mu(\beta) - q^{(i_0-1)(1-\beta)}(\mu(\beta)+1) &= \mu(\beta)\left(1 - q^{(i_0-1)(1-\beta)}\right) - q^{(i_0-1)(1-\beta)}\\ &> 1 - 2q^{(i_0-1)(1-\beta)}\\ &> -1.\end{aligned}$$

Hence we have

$$1 + \gamma_j \phi_{\mathrm{wal},\beta} \geqslant 0$$

for $\gamma_j \leqslant 1$.

For simplification reasons we define $\gamma_j' := \min(\gamma_j, 1)$, $1 \leqslant j \leqslant s$, and $\boldsymbol{\gamma}' = (\gamma_1', \ldots, \gamma_s')$. Thus, $K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}'} \geqslant 0$.

It is easy to see that

$$\|f\|^2_{\mathrm{wal},s,\boldsymbol{\gamma}'} = \sum_{\mathbf{k}\in\mathbb{N}_0^s} r(\beta, \boldsymbol{\gamma}', \mathbf{k})^{-1} |\hat{f}(\mathbf{k})|^2 \geqslant \sum_{\mathbf{k}\in\mathbb{N}_0^s} r(\beta, \boldsymbol{\gamma}, \mathbf{k})^{-1} |\hat{f}(\mathbf{k})|^2 = \|f\|^2_{\mathrm{wal},s,\boldsymbol{\gamma}},$$

for any $f \in \mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}'}$, since $r(\beta, \boldsymbol{\gamma}, \mathbf{k})$ increases with respect to the entries of $\boldsymbol{\gamma}$, as we know from Definition 2.22 and Equation (4). This in turn implies that the closed unit ball of $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}'}$ is contained in the closed unit ball of $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$. For the worst-case error we therefore have

$$e^2_{n,s}(\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}'}) \leqslant e^2_{n,s}(\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}),$$

where $e_{n,s}(\mathscr{H}_{\mathrm{wal},s,\beta,\tilde{\boldsymbol{\gamma}}})$ denotes the worst-case error in the space $\mathscr{H}_{\mathrm{wal},s,\beta,\tilde{\boldsymbol{\gamma}}}$, for an arbitrary but fixed QMC-rule $Q_{n,s}$.

For any point set $\{\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}\} \subseteq [0,1)^s$ we obtain

$$
\begin{aligned}
e_{n,s}^2(\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}'}) &\overset{(22)}{=} -1 + \frac{1}{n^2} \sum_{h,i=0}^{n-1} K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}'}(\mathbf{x}_h, \mathbf{x}_i) \\
&\geqslant -1 + \frac{1}{n^2} \sum_{h=0}^{n-1} K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}'}(\mathbf{x}_h, \mathbf{x}_h) \\
&\overset{(13)}{=} -1 + \frac{1}{n^2} \sum_{h=0}^{n-1} \sum_{\mathbf{k} \in \mathbb{N}_0^s} r(\beta, \boldsymbol{\gamma}', \mathbf{k}) \, \mathrm{wal}_{\mathbf{k}}(\mathbf{x}_h) \overline{\mathrm{wal}_{\mathbf{k}}(\mathbf{x}_h)} \\
&\overset{\text{Prop. 2.10}}{=} -1 + \frac{1}{n^2} \sum_{h=0}^{n-1} \sum_{\mathbf{k} \in \mathbb{N}_0^s} r(\beta, \boldsymbol{\gamma}', \mathbf{k}) \, \mathrm{wal}_{\mathbf{k}}(\mathbf{0}) \\
&= -1 + \frac{1}{n} \sum_{\mathbf{k} \in \mathbb{N}_0^s} r(\beta, \boldsymbol{\gamma}', \mathbf{k}) \\
&= -1 + \frac{1}{n} \prod_{j=1}^{s} \sum_{k=0}^{\infty} r(\beta, \gamma_j', k) \\
&\overset{\text{Lemma 2.16}}{=} -1 + \frac{1}{n} \prod_{j=1}^{s} \left(1 + \gamma_j' \mu(\beta)\right).
\end{aligned}
$$

□

This already suffices to derive necessary conditions for tractability and strong tractability of integration in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$.

Corollary 4.21.

(i) Multivariate integration in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ is strongly QMC-tractable if and only if $\sum_{j=1}^{\infty} \gamma_j < \infty$.

(ii) Multivariate integration in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ is QMC-tractable if and only if $\limsup_{s\to\infty} \sum_{j=1}^{s} \gamma_j / \log s < \infty$.

(cf. [6, Corollary 1])

Proof. The *if*-part has already been shown in Corollary 4.16 for (i) and in Corollary 4.18 for (ii).

So, we assume that multivariate integration in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ is strongly QMC-tractable, i.e. $n_{\min}(\epsilon, s) \leqslant c\epsilon^{-a}$ for any $\epsilon \in (0,1)$ and some constants $c, a \geqslant 0$, both independent of s. In any case, it follows from Theorem 4.20 that

$$n_{\min}(\epsilon, s) \geqslant \frac{\prod_{j=1}^{s} (1 + \min(\gamma_j, 1)\mu(\beta))}{1+\epsilon^2}, \tag{33}$$

(cf. [6, p. 171]).

Now, we will proceed in a similar fashion as H. I. Sloan and H. Woźniakowski did in [18, p. 715]. By definition of $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ we know, that $(\gamma_j)_{j\in\mathbb{N}}$ is a sequence of positive and non-increasing real numbers. Thus, there either exists a γ_0 such that $\gamma_j \geqslant \gamma_0 > 0$ for all $j \geqslant 1$ or the sequence converges to zero. First, suppose the sequence $(\gamma_j)_{j\in\mathbb{N}}$ is uniformly bounded from below by, say, $\gamma_0 > 0$. Then (33) can be estimated further

$$n_{\min}(\epsilon, s) \geqslant \frac{(1 + \min(\gamma_0, 1)\mu(\beta))^s}{1+\epsilon^2}.$$

Hence, $n_{\min}$ grows exponentially with s, which contradicts tractability. Thus, we may restrict ourselves to the case where $\lim_{j\to\infty} \gamma_j = 0$. Additionally, for contradiction we assume $\sum_{j=1}^{\infty} \gamma_j = \infty$. In this case we can rewrite (33) as follows:

$$n_{\min}(\epsilon, s) \geqslant \frac{\exp\left(\sum_{j=1}^{s} \log\left(1 + \min(\gamma_j, 1)\mu(\beta)\right)\right)}{1+\epsilon^2}.$$

If we now manage to show that $\sum_{j=1}^{\infty} \log\left(1 + \min(\gamma_j, 1)\mu(\beta)\right) = \infty$ we can deduce that $n_{\min}(\epsilon, s)$ tends towards infinity as $s \to \infty$, which is clearly a contradiction to strong tractability.

But this is indeed the case as can be found in the proof of [8, Theorem 4], for instance:

$$\sum_{j=1}^{\infty} \gamma_j < \infty \Longleftrightarrow \sum_{j=1}^{\infty} \log(1+\gamma_j) < \infty.$$

The implication from left to right follows immediately from the fact that $\log(1+\gamma_j) \leqslant \gamma_j$. On the other hand, we know that $(\gamma_j)_{j\in\mathbb{N}}$ is a non-increasing sequence, which means there exists a constant $C > 0$ with $\gamma_j \leqslant C$ for all $j \in \mathbb{N}$. Therefore we can find another constant $d = d(C) > 0$ such that $dx \leqslant \log(1+x)$ for any $x \in [0, C]$ and consequently

$$\sum_{j=1}^{\infty} \gamma_j \leqslant \frac{1}{d} \sum_{j=1}^{\infty} \log(1+\gamma_j) < \infty,$$

which completes the proof of the first part.

It is left to show is that QMC-tractability in $\mathscr{H}_{\text{wal},s,\beta,\boldsymbol{\gamma}}$ implies

$$\limsup_{s\to\infty} \frac{\sum_{j=1}^{s}\gamma_j}{\log s} < \infty.$$

Following [18, p. 715] again, we already know that $\lim_{j\to\infty}\gamma_j = 0$ is a necessary condition for tractability. Rewriting the numerator of the right handside in (33) gives

$$\prod_{j=1}^{s}\big(1+\min(\gamma_j,1)\mu(\beta)\big) = s^{\sum_{j=1}^{s}\log_s\left(1+\min(\gamma_j,1)\mu(\beta)\right)} = s^{\sum_{j=1}^{s}\frac{\log(1+\min(\gamma_j,1)\mu(\beta))}{\log s}}.$$

Under the assumption that $\limsup_{s\to\infty}\sum_{j=1}^{s}\gamma_j/\log(s) = \infty$ and using the same argument as in the proof of the first part we can now conclude that $n_{\min}(\epsilon,s)$ grows faster than any power of s, contradicting QMC-tractability. □

Finally, we investigate what happens to the worst-case error whenever β approaches 1.

Corollary 4.22. *Let $Q_{n,s}$ be an arbitrary QMC-algorithm and let $e(Q_{n,s},\mathscr{H}_{\text{wal},s,\beta,\boldsymbol{\gamma}})$ denote the worst-case error for integration in $\mathscr{H}_{\text{wal},s,\beta,\boldsymbol{\gamma}}$. Then we have*

$$\lim_{\beta\to 1^+} e(Q_{n,s},\mathscr{H}_{\text{wal},s,\beta,\boldsymbol{\gamma}}) = \infty.$$

(cf. [6, Theorem 5])

Proof. (Taken from [6, Theorem 5]).

From Theorem 4.20 we know that

$$e^2(Q_{n,s},\mathscr{H}_{\text{wal},s,\beta,\boldsymbol{\gamma}}) \geqslant -1 + \frac{1}{n}\prod_{j=1}^{s}\big(1+\min(\gamma_j,1)\mu(\beta)\big)$$

and together with

$$\lim_{\beta\to 1^+}\mu(\beta) = \lim_{\beta\to 1^+}\frac{q^\beta(q-1)}{q^\beta - q} = \infty$$

we obtain the desired result. □

5 Construction algorithms for digital (t, m, s)-nets over $\mathbb{F}_q$

Although we have the existence of digital (t, m, s)-nets over $\mathbb{F}_q$ with a rather favourable behavior with respect to the worst-case error at hand, which may, in some cases, even exploit (strong) tractability, we still have not yet solved the problem of finding such. To this end we will intoduce algorithms for constructing generating matrices using formal Laurent series over $\mathbb{F}_q$ and investigate the performance of the resulting digital nets concerning integration in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$.

As it was also done in [4, p. 1898], we begin by defining the field of formal Laurent series over the finite field $\mathbb{F}_q$, i.e.

$$\mathbb{F}_q((x^{-1})) := \left\{ \sum_{l=w}^{\infty} t_l x^{-l} : \ w \in \mathbb{Z} \text{ and } t_l \in \mathbb{F}_q \text{ for all } l \geqslant w \right\},$$

and denote the set of all polynomials over $\mathbb{F}_q$ by $\mathbb{F}_q[x]$. The first types of digital nets we are drawing our attention to are so-called polynomial lattice point sets, which were first introduced by H. Niederreiter in [12] (see also Section 4.4 in [14]).

5.1 Polynomial lattice point sets

Definition 5.1 (Polynomial lattice rule)**.** Let $\mathfrak{p} \in \mathbb{F}_q[x]$ with $\deg \mathfrak{p} = m \in \mathbb{N}$. Additionally, we consider $\mathfrak{q}_1, \ldots, \mathfrak{q}_s \in \mathbb{F}_q[x]$ and the Laurent series expansions

$$\frac{\mathfrak{q}_j(x)}{\mathfrak{p}(x)} = \sum_{l=w_j}^{\infty} u_l^{(j)} x^{-l} \in \mathbb{F}_q((x^{-1}))$$

with $w_j \leqslant 1$, for $1 \leqslant j \leqslant s$. Furthermore, we set

$$c_{i,r}^{(j)} = u_{i+r}^{(j)} \in \mathbb{F}_q,$$

where $1 \leqslant i \leqslant m$, $0 \leqslant r \leqslant m-1$ and define

$$C_j := \left(c_{i,r}^{(j)}\right)_{\substack{i=1,\ldots,m \\ r=0,\ldots,m-1}} = \begin{pmatrix} u_1^{(j)} & u_2^{(j)} & \cdots & u_m^{(j)} \\ u_2^{(j)} & & \iddots & \vdots \\ \vdots & \iddots & & \vdots \\ u_m^{(j)} & \cdots & \cdots & u_{2m-1}^{(j)} \end{pmatrix} \tag{34}$$

for all $1 \leqslant j \leqslant s$. The matrices $C_1, \ldots, C_s$ are now used to construct a digital (t, m, s)-net over $\mathbb{F}_q$ as described in Definition 3.4, forming a so-called *polynomial lattice point set.* We denote this set by $\mathcal{S}_{\mathfrak{p}}(\mathfrak{q})$, where $\mathfrak{q} = (\mathfrak{q}_1, \ldots, \mathfrak{q}_s)$, and call a QMC-rule employing $\mathcal{S}_{\mathfrak{p}}(\mathfrak{q})$ as sample points a *polynomial lattice rule.*

(cf. [4, Definition 2.3, Remark 2.4])

Remark 5.2. Since $u_1^{(j)}$ is the entry with the lowest index in C_j, it suffices to merely consider such polynomials $\mathfrak{q}_j$ with $\deg(\mathfrak{q}_j) \leqslant m - 1$, $1 \leqslant j \leqslant s$, (cf. [5, Remark 10.12]).

In the case where q is a prime number, the construction principle of polynomial lattice point sets can be held considerably simpler.

Theorem 5.3. *Let q be a prime, $\mathfrak{p} \in \mathbb{F}_q[x]$ with $\deg(\mathfrak{p}) = m \in \mathbb{N}$ and $\mathfrak{q} = (\mathfrak{q}_1, \ldots, \mathfrak{q}_s) \in \mathbb{F}_q^s[x]$. We define the map $v_m \colon \mathbb{F}_q((x^{-1})) \to [0, 1)$ by*

$$v_m \left(\sum_{l=w}^{\infty} t_l x^{-l} \right) := \sum_{l=\max(1,w)}^{m} t_l q^{-l}.$$

As, in the prime case, $\mathbb{F}_q$ corresponds to $\mathbb{Z}_q$, we may choose $\varphi_1 = id$ and hence, with every $0 \leqslant h < q^m$ with q-adic expansion $h = h_1 + h_2 q + \cdots + h_m q^{m-1}$ we can associate the polynomial

$$\mathfrak{h}(x) = h_1 + h_2 x + \cdots + h_m x^{m-1} \in \mathbb{F}_q[x].$$

Then, the polynomial lattice point set $\mathcal{S}_p(\mathfrak{q})$ is given by the points

$$\mathbf{x}_h = \left(v_m \left(\frac{\mathfrak{h}(x)\mathfrak{q}_1(x)}{\mathfrak{p}(x)} \right), \ldots, v_m \left(\frac{\mathfrak{h}(x)\mathfrak{q}_s(x)}{\mathfrak{p}(x)} \right) \right),$$

$0 \leqslant h < q^m$.

(cf. [4, Remark 2.7])

Proof. In accordance to Definition 5.1 we expand

$$\frac{\mathfrak{q}_j(x)}{\mathfrak{p}(x)} = \sum_{l=w_j}^{\infty} u_l^{(j)} x^{-l}$$

for all $1 \leqslant j \leqslant s$. For the evaluation of v_m applied to the following fraction we adhere to the proof of [5, Theorem 10.5]. For an arbitrary $0 \leqslant h < q^m$ with q-adic expansion $h = \sum_{k=0}^{m-1} h_{k+1} q^k$ we get

$$\frac{\mathfrak{h}(x)\mathfrak{q}_j(x)}{\mathfrak{p}(x)} = \left(\sum_{l=w_j}^{\infty} u_l^{(j)} x^{-l} \right) \left(\sum_{k=0}^{m-1} h_{k+1} x^k \right) = \sum_{l=w_j}^{\infty} u_l^{(j)} \sum_{k=0}^{m-1} h_{k+1} x^{k-l}.$$

We substitute $r := l - k$ and set $u_i^{(j)} = 0$ for $1 \leqslant i < w_j$, if existent. Since v_m only considers the truncated polynomial where $1 \leqslant r \leqslant m$, we obtain

$$v_m \left(\frac{\mathfrak{h}(x)\mathfrak{q}_j(x)}{\mathfrak{p}(x)} \right) = v_m \left(\sum_{r=1}^{m} x^{-r} \sum_{l=0}^{m-1} u_{r+l}^{(j)} h_{l+1} \right) = \sum_{r=1}^{m} q^{-r} \sum_{l=0}^{m-1} u_{r+l}^{(j)} h_{l+1}.$$

It needs to be added, that the sum over l is evaluated in $\mathbb{F}_q$.

Let $\mathbf{c}_{j,k} = (u_k^{(j)}, \ldots, u_{k+m-1}^{(j)})$ denote the kth row of the generating matrix C_j. Thus, using the fact that C_j is symmetric, we have

$$y_j^{(k)}(h) = \mathbf{c}_{j,k}(h_1, \ldots, h_m)^\top = \sum_{l=0}^{m-1} u_{k+l}^{(j)} h_{l+1}.$$

As the jth component of the point $\mathbf{x}_h$ is given by $\sum_{k=1}^{m} y_j^{(k)}(h) q^{-k}$ we are finished. $\square$

For some choices of the polynomials involved we can even make assertions concerning the distribution of the point set $\mathcal{S}_{\mathfrak{p}}(\mathbf{\mathfrak{q}})$ itself, as it will be shown in the next theorem. To do so, however, we require the result of the following lemma which was given by R. Lidl and H. Niederreiter in [11].

Lemma 5.4. *Let* $\mathfrak{q}_j, \mathfrak{p} \in \mathbb{F}_q[x]$ *with* $\deg \mathfrak{p} = m \in \mathbb{N}$ *and* $\gcd(\mathfrak{q}_j, \mathfrak{p}) = 1$. *Then, the matrix* $C_j \in \mathbb{F}_q^{m \times m}$ *obtained by the methods from Definition 5.1 is regular.*

Proof. See [11, Theorem 6.75]. $\square$

Theorem 5.5. *Let* $\mathfrak{p} \in \mathbb{F}_q[x]$ *with* $\deg(\mathfrak{p}) = m \in \mathbb{N}$. *Furthermore, let* $\mathfrak{q}_1, \ldots, \mathfrak{q}_s \in \mathbb{F}_q[x]$ *such that* $\gcd(\mathfrak{q}_j, \mathfrak{p}) = 1$ *for all* $1 \leqslant j \leqslant s$. *By* $x_h^{(j)}$ *we denote the* j*th coordinate of the point* $\mathbf{x}_h \in \mathcal{S}_{\mathfrak{p}}(\mathfrak{q})$, $0 \leqslant h < q^m$. *Then, for every* $1 \leqslant j \leqslant s$ *the point set* $\{x_0^{(j)}, \ldots, x_{q^m-1}^{(j)}\}$ *is a* $(0, m, 1)$*-net over* $\mathbb{F}_q$.

(cf. [4, Remark 2.5])

Proof. Let $1 \leqslant j \leqslant s$ be a fixed integer and let

$$E := \left[\frac{a}{q^m}, \frac{a+1}{q^m}\right),$$

where $0 \leqslant a < q^m$, be an arbitrary one-dimensional elementary interval in base q of order m. We need to show that there exists exactly one $h \in \{0, \ldots, q^m - 1\}$ such that $x_h^{(j)} \in E$. If we consider the q-adic expansion of a, i.e.

$$a = a_1 + a_2 q + \cdots + a_m q^{m-1},$$

we immediately notice that

$$x \in E \iff x \text{ has the } q\text{-adic expansion } x = \frac{a_m}{q} + \cdots + \frac{a_1}{q^m} + \frac{\xi_{m+1}}{q^{m+1}} + \cdots$$

with suitable coefficients $0 \leqslant \xi_k < q$, $k > m$. Thus, to find a unique $h = h_1 + \cdots + h_m q^{m-1}$ with the desired property we need to solve the system of linear equations

$$C_j \begin{pmatrix} \varphi_1(h_1) \\ \vdots \\ \varphi_1(h_m) \end{pmatrix} = \begin{pmatrix} \varphi_1(a_m) \\ \vdots \\ \varphi_1(a_1) \end{pmatrix}.$$

Lemma 5.4 secures the existence of exactly one solution and togehter with the fact that φ_1 is bijective the result follows. $\square$

Apart from deciding on which polynomials to choose for the construction of a polynomial lattice point set, the only tedious task left is to determine the coefficients of the Laurent series expansion. This, however, is facilitated by the identities given in the following theorem.

Theorem 5.6. *Let $m \in \mathbb{N}$ and $\mathfrak{p}, \mathfrak{q} \in \mathbb{F}_q[x]$, where*

$$\mathfrak{p}(x) = x^m + p_1 x^{m-1} + \cdots + p_{m-1}x + p_m$$

and

$$\mathfrak{q}(x) = q_1 x^{m-1} + \cdots + q_{m-1}x + q_m.$$

Then, the coefficients $u_l \in \mathbb{F}_q$, $l \geqslant 1$ from the Laurent series expansion

$$\frac{\mathfrak{q}(x)}{\mathfrak{p}(x)} = \sum_{l=1}^{\infty} u_l x^{-l}$$

can be retrieved for $l \leqslant m$ from the linear system of equations

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ p_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ p_{m-1} & \cdots & p_1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{pmatrix}$$

and from the recursion

$$0 = u_l + u_{l-1}p_1 + u_{l-2}p_2 + \cdots + u_{l-m}p_m$$

for $l > m$.

(cf. [4, Proposition 2.6])

Proof. First of all, we rewrite the occuring polynomials into a closed sum, giving $\mathfrak{p}(x) = \sum_{i=0}^{m} p_{m-i}x^i$ and $\mathfrak{q}(x) = \sum_{j=0}^{m-1} q_{m-j}x^j$, where $p_0 := 1$, and consider the equation

$$\begin{aligned} \sum_{j=0}^{m-1} q_{m-j}x^j = \mathfrak{q}(x) = \mathfrak{p}(x) \sum_{l=1}^{\infty} u_l x^{-l} \\ = \sum_{l=1}^{\infty} u_l \sum_{i=0}^{m} p_{m-i} x^{i-l} \\ = \sum_{l=1}^{\infty} u_l \sum_{r=-l}^{m-l} p_{m-r-l} x^r. \end{aligned}$$

For simplicity reasons we set $p_i = 0$ for $i \notin \{0, \dots m\}$. Comparing the coefficients of the various powers of x, as suggested in the proof of [4, Proposition 2.6], then leads to

$$\begin{aligned} q_{m-j} &= \sum_{l=1}^{\infty} u_l p_{m-j-l} \text{ for } 0 \leqslant j \leqslant m-1 \\ \Longleftrightarrow \quad q_k &= \sum_{l=1}^{\infty} u_l p_{k-l} \text{ for } 1 \leqslant k \leqslant m. \end{aligned} \tag{35}$$

Thus, for any $k \in \{1, \dots, m\}$ we have (after eliminating the cases where $p_{k-l} = 0$)

$$q_k = \sum_{l=1}^{k} u_l p_{k-l},$$

which, most certainly, corresponds to the linear system of equations given in the claim of this theorem. Similarly, we see from (35) that

$$0 = \sum_{l=k-m}^{k} u_l p_{k-l}$$

for $k > m$, which proves the second assertion. $\square$

Before we proceed further, we introduce some notation. In the case where q is a prime number similar definitions can be found in [4, pp. 1900 and 1904f.].

For $1 \leqslant k < q^m$ with q-adic expansion $k = \kappa_1 + \kappa_2 q + \cdots + \kappa_m q^{m-1}$ we define the polynomial $\mathfrak{k}(x) = \varphi_1(\kappa_1) + \varphi_1(\kappa_2)x + \cdots + \varphi_1(\kappa_m)x^{m-1} \in \mathbb{F}_q[x]$. Furthermore, we introduce the mapping $\mathrm{tr}_m \colon \mathbb{N}_0 \to \mathbb{Z}_{q^m}$ with

$$\mathrm{tr}_m \left(\sum_{i=0}^{\infty} \kappa_{i+1} q^i \right) := \kappa_1 + \kappa_2 q + \cdots + \kappa_m q^{m-1},$$

where $0 \leqslant \kappa_i < q$ for all $i \in \mathbb{N}$. Similarly to the above case, we associate $\mathrm{tr}_m(k)$, $k = \sum_{i=0}^{\infty} \kappa_{i+1} q^i$, with the polynomial

$$\mathrm{tr}_m(\mathfrak{k}) = \varphi_1(\kappa_1) + \varphi_1(\kappa_2)x + \cdots + \varphi_1(\kappa_m)x^{m-1} \in \mathbb{F}_q[x].$$

For $\mathbf{k} \in \mathbb{N}_0^s$ we define $\mathrm{tr}_m(\mathbf{k})$ and $\mathrm{tr}_m(\mathfrak{k})$ componentwise.

Moreover, for $\mathfrak{p} = (\mathfrak{p}_1, \ldots, \mathfrak{p}_s)$ and $\mathfrak{q} = (\mathfrak{q}_1, \ldots, \mathfrak{q}_s)$, both in $\mathbb{F}_q^s[x]$, we set

$$\mathfrak{p} \cdot \mathfrak{q} := \sum_{j=1}^{s} \mathfrak{p}_j \mathfrak{q}_j \in \mathbb{F}_q[x].$$

Finally, we define the set of all non-zero polynomials over $\mathbb{F}_q$ with degree less than m by

$$G_{q,m} := \{\mathfrak{k} \in \mathbb{F}_q[x] \backslash \{0\} : \ \deg(\mathfrak{k}) < m\}$$

and its s-dimensional analogon by the s-fold cartesian product

$$G_{q,m}^s := \prod_{j=1}^{s} G_{q,m}.$$

We now draw our attention to the square worst-case error again. In Theorem 4.12 we have shown that for any s-tuple of generating matrices $(C_1, \ldots, C_s)$ over $\mathbb{F}_q$ we can compute the worst-case error for the corresponding digital net by evaluating the sum

$$e^2_{q^m,s}(C_1, \ldots, C_s) = \sum_{\mathbf{k} \in \mathcal{D} \backslash \{\mathbf{0}\}} r(\beta, \boldsymbol{\gamma}, \mathbf{k}),$$

where the dual net is given by

$$\mathcal{D} = \left\{ (k_1, \ldots, k_s) \in \mathbb{N}_0^s : \ \sum_{j=1}^{s} C_j^\top \varphi(k_j) = \mathbf{0} \right\}.$$

It hardly comes as a surprise that we obtain a very similar result for polynomial lattice point sets.

Lemma 5.7. *Let* $\mathfrak{p} \in \mathbb{F}_q[x]$ *with* $\deg(\mathfrak{p}) = m \in \mathbb{N}$ *and* $\mathfrak{q} = (\mathfrak{q}_1, \ldots, \mathfrak{q}_s) \in \mathbb{F}_q^s[x]$*. Then, for a QMC-rule employing the polynomial lattice point set* $\mathcal{S}_\mathfrak{p}(\mathfrak{q})$*, the worst-case error for integration in* $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ *can be calculated via*

$$e^2_{q^m,s}(\mathcal{S}_\mathfrak{p}(\mathfrak{q})) = \sum_{\mathbf{k} \in \mathcal{D}^*_{\mathfrak{p},\mathfrak{q}}} r(\beta, \boldsymbol{\gamma}, \mathbf{k}).$$

Here, the dual net $\mathcal{D}_{\mathfrak{p},\mathfrak{q}}$ *is defined as*

$$\mathcal{D}_{\mathfrak{p},\mathfrak{q}} := \{\mathbf{k} \in \mathbb{N}_0^s\colon \ \mathrm{tr}_m(\mathfrak{k}) \cdot \mathfrak{q} \equiv 0 \pmod{\mathfrak{p}}\}$$

and $\mathcal{D}^*_{\mathfrak{p},\mathfrak{q}} := \mathcal{D}_{\mathfrak{p},\mathfrak{q}} \backslash \{\mathbf{0}\}$*, where for two polynomials* $\mathfrak{r}, \mathfrak{s} \in \mathbb{F}_q[x]$ *we interpret* $\mathfrak{r} \equiv 0 \pmod{\mathfrak{s}}$ *as* $\mathfrak{s}$ *divides* $\mathfrak{r}$ *in* $\mathbb{F}_q[x]$*.*

(cf. [4, Lemma 4.1] and [5, p. 300])

Proof. For all $1 \leqslant j \leqslant s$ let

$$\frac{\mathfrak{q}_j}{\mathfrak{p}} = \sum_{l=w_j}^{\infty} u_l^{(j)} x^{-l} \in \mathbb{F}_q((x^{(-1)}))$$

and let $C_1, \ldots, C_s \in \mathbb{F}_q^{m\times m}$ be the generating matrices obtained by (34). Due to Theorem 4.12 it suffices to show that

$$\sum_{j=1}^{s} C_j^\top \varphi(k_j) = \mathbf{0} \Longleftrightarrow \operatorname{tr}_m(\mathfrak{k}) \cdot \mathfrak{q} \equiv 0 \pmod{\mathfrak{p}}$$

for every $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s \backslash \{\mathbf{0}\}$, as indicated in the proof of [4, Lemma 4.1].

To this end, we follow the proof of [5, Lemma 10.6]. So, first of all, we intend to find a formula for the coefficient of x^{-r}, $r \in \mathbb{N}$, in $(\operatorname{tr}_m(\mathfrak{k}) \cdot \mathfrak{q})/\mathfrak{p}$. For every $1 \leqslant j \leqslant s$ we consider the polynomial $\operatorname{tr}_m(\mathfrak{k}_j)(x) = \varphi_1(\kappa_{j,1}) + \varphi_1(\kappa_{j,2})x + \cdots + \varphi_1(\kappa_{j,m})x^{m-1} \in \mathbb{F}_q[x]$, where $\kappa_{j,i}$ denotes the ith q-adic digit of k_j. We have

$$\begin{aligned}
\frac{\operatorname{tr}_m(\mathfrak{k}_j)\mathfrak{q}_j}{\mathfrak{p}} &= \left(\sum_{i=0}^{m-1} \varphi_1(\kappa_{j,i+1}) x^i\right) \left(\sum_{l=w_j}^{\infty} u_l^{(j)} x^{-l}\right) \\
&= \sum_{i=0}^{m-1} \varphi_1(\kappa_{j,i+1}) \sum_{l=w_j}^{\infty} u_l^{(j)} x^{i-l} \\
&= \sum_{i=0}^{m-1} \varphi_1(\kappa_{j,i+1}) \sum_{r=w_j-i}^{\infty} u_{r+i}^{(j)} x^{-r}.
\end{aligned}$$

Therefore, the coefficient of x^{-r}, $r \in \mathbb{N}$, in the above fraction is given by

$$\sum_{i=0}^{m-1} \varphi_1(\kappa_{j,i+1}) u_{r+i}^{(j)}$$

and hence the coefficient of x^{-r} in $(\operatorname{tr}_m(\mathfrak{k}) \cdot \mathfrak{q})/\mathfrak{p}$ is

$$\sum_{j=1}^{s} \sum_{i=0}^{m-1} \varphi_1(\kappa_{j,i+1}) u_{r+i}^{(j)}.$$

By taking a closer look on the condition for $(k_1, \ldots, k_s)$ being an element of $\mathcal{D}$ we find that

$$C_1^\top \varphi(k_1) + \cdots + C_s^\top \varphi(k_s) = \mathbf{0} \Longleftrightarrow \forall 1 \leqslant r \leqslant m\colon\ \sum_{j=1}^{s} \sum_{i=0}^{m-1} \varphi_1(\kappa_{j,i+1}) u_{r+i}^{(j)} = 0.$$

This, however, is equivalent to the coefficients of x^{-r} in $(\mathrm{tr}_m(\mathfrak{k}_j)\mathfrak{q}_j)/\mathfrak{p}$ being zero for all $1 \leqslant r \leqslant m$, i.e.

$$\frac{1}{\mathfrak{p}}\mathrm{tr}_m(\mathfrak{k}) \cdot \mathfrak{q} = \mathfrak{g} + L \tag{36}$$

for some $\mathfrak{g} \in \mathbb{F}_q[x]$ and some $L = \sum_{k=m+1}^{\infty} f_k x^{-k} \in \mathbb{F}((x^{-1}))$. Note that this sum starts at $k = m+1$. Rearranging this equation yields

$$\mathrm{tr}_m(\mathfrak{k}) \cdot \mathfrak{q} - \mathfrak{g}\mathfrak{p} = L\mathfrak{p}.$$

We observe that the highest power of x in L is $-(m+1)$ at most and $\mathfrak{p}$ has degree m, whereas on the left handside we have an element of $\mathbb{F}_q[x]$, leaving $L = 0$ as the only possibility, which in turn means that $\mathfrak{p}$ divides $\mathrm{tr}_m(\mathfrak{k}) \cdot \mathfrak{q}$. Thus, we can say that (36) holds if and only if $\mathrm{tr}_m(\mathfrak{k}) \cdot \mathfrak{q} \equiv 0 \pmod{\mathfrak{p}}$. □

We now consider the case where $\mathfrak{p}$ is irreducible over $\mathbb{F}_q$. By definition we have that $\mathbf{k} = (k_1, \ldots, k_s) \in \mathcal{D}_{\mathfrak{p},\mathfrak{q}}$ iff

$$\mathrm{tr}_m(\mathfrak{k}) \cdot \mathfrak{q} = \mathrm{tr}_m(\mathfrak{k}_1)\mathfrak{q}_1 + \cdots + \mathrm{tr}_m(\mathfrak{k}_s)\mathfrak{q}_s \equiv 0 \pmod{\mathfrak{p}}.$$

Since $\mathfrak{p}$ is irreducible, it follows that $\gcd(\mathfrak{q}_1, \mathfrak{p}) = 1$ in $\mathbb{F}_q[x]$ whenever $\deg(\mathfrak{q}_1) < \deg(\mathfrak{p})$ and $\mathfrak{q}_1 \neq 0$. This, however, is no restriction due to Remark 5.2. Hence, one can always find $\mathfrak{q}_1^* \in \mathbb{F}_q[x]$ such that $\mathfrak{q}_1^*\mathfrak{q}_1 \equiv 1 \pmod{\mathfrak{p}}$. Thus, whenever we choose $\mathfrak{p}$ irreducible over $\mathbb{F}_q[x]$ and we want to make assertions involving the dual net $\mathcal{D}_{\mathfrak{p},\mathfrak{q}}$ or – due to Lemma 5.7 – the worst-case error, we can restrict ourselves to the case where $\mathfrak{q}_1 = 1$, (cf. [5, Remark 10.10]).

5.1.1 The component-by-component construction

In Section 4.2 we have shown several existence results for digital (t, m, s)-nets satisfying a certain error bound or exploiting (strong) tractability under certain conditions (see, for instance, Theorem 4.15 and Corollaries 4.18 and 4.16).

Lemma 5.7 together with Remark 5.2 and the above discussion already provide enough information to state an executable routine to find a digital net for which the worst-case error behaves rather favourably. The algorithm works as follows:

Algorithm 5.8 (Component-by-component construction)**.** Choose an irreducible polynomial $\mathfrak{p} \in \mathbb{F}_q[x]$ with $\deg(\mathfrak{p}) = m \in \mathbb{N}$ and let all parameters necessary to define the weighted Hilbert space $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ be given.

1. $\mathfrak{q}_1 := 1$.
2. For $d = 2, \ldots, s$ find $\mathfrak{q}_d \in G_{q,m}$ which minimizes $e^2(\mathcal{S}_{\mathfrak{p}}(\mathfrak{q}_1, \ldots, \mathfrak{q}_d))$ over $G_{q,m}$.
3. Return $\boldsymbol{\mathfrak{q}} = (\mathfrak{q}_1, \ldots, \mathfrak{q}_s)$.

(cf. [4, Algorithm 4.3])

It is to mention that this algorithm terminates after finitely many steps as $|G_{q,m}| = q^m - 1$, (cf. [4, p. 1905]).

We can now estimate the worst-case error for integration in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ using a polynomial lattice rule obtained by the above algorithm at every intermediate step.

Theorem 5.9. *Let $\mathfrak{p}$ be an irreducible polynomial over $\mathbb{F}_q$ with* $\deg(\mathfrak{p}) = m \in \mathbb{N}$ *and assume that $\boldsymbol{\mathfrak{q}}^* := (\mathfrak{q}_1^*, \ldots, \mathfrak{q}_s^*) \in G_{q,m}^s$ are the polynomials generated by the component-by-component (CBC) construction. Then, for all $d \in \{1, \ldots, s\}$ and for all $1/\beta < \lambda \leqslant 1$ the following inequality holds:*

$$e^2_{q^m,d}(\mathcal{S}_{\mathfrak{p}}(\mathfrak{q}_1^*, \ldots, \mathfrak{q}_d^*)) \leqslant \frac{1}{(q^m - 1)^{\frac{1}{\lambda}}} \prod_{j=1}^{d} \left(1 + \mu(\beta\lambda)\gamma_j^{\lambda}\right)^{\frac{1}{\lambda}},$$

where μ is defined as in Lemma 2.16.

(cf. [4, Theorem 4.4])

Proof. We recall that in (31) it has already been shown that $r(\beta, \boldsymbol{\gamma}, \mathbf{k})^{\lambda} = r(\beta\lambda, \boldsymbol{\gamma}^{\lambda}, \mathbf{k})$.

We begin with the proof by looking at the congruence

$$\mathrm{tr}_m(\mathfrak{k})\mathfrak{q}_1^* = \mathrm{tr}_m(\mathfrak{k}) \equiv 0 \pmod{\mathfrak{p}}, \tag{37}$$

where $\mathfrak{k}$ is the associated polynomial to $k \in \mathbb{N}$. It is clear that every solution of (37) has to be a multiple of q^m, i.e. $k = lq^m$, $l \in \mathbb{N}$, as $\deg(\mathfrak{p}) = m > \deg(\mathrm{tr}_m(\mathfrak{k}))$. Hence, for $d = 1$ we obtain

$$e^2_{q^m,1}(\mathcal{S}_{\mathfrak{p}}(1)) \overset{\text{Lemma 5.7}}{=} \sum_{k \in \mathcal{D}^*_{\mathfrak{p},1}} r(\beta, \gamma_1, k) = \sum_{l=1}^{\infty} r(\beta, \gamma_1, lq^m).$$

Let $1/\beta < \lambda \leqslant 1$. Applying Jensen's inequality (see (29)) and using the identity given in the proof of Lemma 4.14, Equation (26) yields

$$\begin{aligned} e^2_{q^m,1}(\mathcal{S}_{\mathfrak{p}}(1)) &\leqslant \left(\sum_{l=1}^{\infty} r(\beta, \gamma_1, lq^m)^{\lambda}\right)^{\frac{1}{\lambda}} \\ &= \left(\sum_{l=1}^{\infty} r(\beta\lambda, \gamma_1^{\lambda}, lq^m)\right)^{\frac{1}{\lambda}} \\ &= \left(\frac{1}{q^{\beta\lambda m}} \gamma_1^{\lambda} \mu(\beta\lambda)\right)^{\frac{1}{\lambda}} \\ &\overset{1<\beta\lambda}{\leqslant} (q^m - 1)^{-\frac{1}{\lambda}} \left(1 + \mu(\beta\lambda)\gamma_1^{\lambda}\right)^{\frac{1}{\lambda}}. \end{aligned}$$

Thus, the assertion is true for $d = 1$.

For the rest of the proof we adhere to the proof of [4, Theorem 4.4]. Let $\mathfrak{q}^*_k := (\mathfrak{q}^*_1, \ldots, \mathfrak{q}^*_k)$ denote the vector consisting of the first k polynomials obtained from Algorithm 5.8 and assume that for some $d \in \{1, \ldots, s-1\}$ we have

$$e^2_{q^m,d}\left(\mathcal{S}_{\mathfrak{p}}(\mathfrak{q}^*_d)\right) \leqslant \frac{1}{(q^m - 1)^{\frac{1}{\lambda}}} \prod_{j=1}^{d} \left(1 + \mu(\beta\lambda)\gamma_j^{\lambda}\right)^{\frac{1}{\lambda}}, \tag{38}$$

$1/\beta < \lambda \leqslant 1$. We now consider $\mathcal{S}_{\mathfrak{p}}(\mathfrak{q}^*_d, \mathfrak{q}_{d+1})$ for some arbitrary $\mathfrak{q}_{d+1} \in G_{q,m}$, where

$$(\mathfrak{q}^*_d, \mathfrak{q}_{d+1}) = (\mathfrak{q}^*_1, \ldots, \mathfrak{q}^*_d, \mathfrak{q}_{d+1}).$$

Using Lemma 5.7 again we find that

$$
\begin{aligned}
e^2_{q^m,d+1}&(\mathcal{S}_{\mathfrak{p}}(\mathfrak{q}^*_d, \mathfrak{q}_{d+1})) = \\
&= \sum_{\mathbf{k}\in\mathcal{D}^*_{\mathfrak{p},(\mathfrak{q}^*_d,\mathfrak{q}_{d+1})}} r(\beta, (\boldsymbol{\gamma}, \gamma_{d+1}), \mathbf{k}) \\
&= \sum_{\substack{(\mathbf{k},k_{d+1})\in\mathbb{N}_0^{d+1}\setminus\{\mathbf{0}\} \\ \mathrm{tr}_m(\mathfrak{k},\mathfrak{k}_{d+1})\cdot(\mathfrak{q}^*_d,\mathfrak{q}_{d+1})\equiv 0 \pmod{\mathfrak{p}}}} r(\beta, \boldsymbol{\gamma}, \mathbf{k})\, r(\beta, \gamma_{d+1}, k_{d+1}) \\
&= \underbrace{\sum_{\substack{\mathbf{k}\in\mathbb{N}_0^d\setminus\{\mathbf{0}\} \\ \mathrm{tr}_m(\mathfrak{k})\cdot\mathfrak{q}^*_d\equiv 0 \pmod{\mathfrak{p}}}} r(\beta, \boldsymbol{\gamma}, \mathbf{k})}_{k_{d+1}=0} \\
&\quad + \underbrace{\sum_{k_{d+1}=1}^{\infty} r(\beta, \gamma_{d+1}, k_{d+1}) \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^d \\ \mathrm{tr}_m(\mathfrak{k})\cdot\mathfrak{q}^*_d\equiv -\mathrm{tr}_m(\mathfrak{k}_{d+1})\mathfrak{q}_{d+1} \pmod{\mathfrak{p}}}} r(\beta, \boldsymbol{\gamma}, \mathbf{k})}_{k_{d+1}\neq 0} \\
&\overset{\text{Lemma 5.7}}{=} e^2_{q^m,d}(\mathcal{S}_{\mathfrak{p}}(\mathfrak{q}^*_d)) + \theta(\mathfrak{q}_{d+1}),
\end{aligned}
\tag{39}
$$

where we define $\theta(\mathfrak{q}_{d+1})$ as the double-sum (i.e. the case where $k_{d+1} \neq 0$) in the above equation. Since we choose $\mathfrak{q}^*_{d+1}$ as a minimizer of $e^2_{q^m,d+1}(\mathcal{S}_{\mathfrak{p}}(\mathfrak{q}^*_d, \cdot))$ over $G_{q,m}$ and as in (39) only θ is dependent on $\mathfrak{q}^*_{d+1}$, it follows that $\theta^\lambda(\mathfrak{q}^*_{d+1}) \leqslant \theta^\lambda(q_{d+1})$ for all $\mathfrak{q}_{d+1} \in G_{q,m}$ and all $\lambda \in (1/\beta, 1]$. Consequently,

$$
\theta(\mathfrak{q}^*_{d+1}) \leqslant \left(\frac{1}{q^m - 1} \sum_{\mathfrak{q}_{d+1}\in G_{q,m}} \theta^\lambda(\mathfrak{q}_{d+1}) \right)^{\frac{1}{\lambda}}. \tag{40}
$$

After applying Jensen's inequality twice we get

$$
\theta^\lambda(\mathfrak{q}_{d+1}) \leqslant \sum_{k_{d+1}=1}^{\infty} r(\beta\lambda, \gamma^\lambda_{d+1}, k_{d+1}) \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^d \\ \mathrm{tr}_m(\mathfrak{k})\cdot\mathfrak{q}^*_d\equiv -\mathrm{tr}_m(\mathfrak{k}_{d+1})\mathfrak{q}_{d+1} \pmod{\mathfrak{p}}}} r(\beta\lambda, \boldsymbol{\gamma}^\lambda, \mathbf{k}). \tag{41}
$$

Averaging $\theta^{\lambda}(\cdot)$ over $G_{q,m}$ yields

$$\begin{aligned}
&\frac{1}{q^m - 1} \sum_{\mathfrak{q}_{d+1} \in G_{q,m}} \theta^{\lambda}(\mathfrak{q}_{d+1}) \\
&\overset{(41)}{\leqslant} \frac{1}{q^m - 1} \sum_{\mathfrak{q}_{d+1} \in G_{q,m}} \Bigg(\sum_{\substack{k_{d+1}=1 \\ q^m | k_{d+1}}}^{\infty} r(\beta\lambda, \gamma_{d+1}^{\lambda}, k_{d+1}) \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^d \\ \mathrm{tr}_m(\mathfrak{k}) \cdot \mathfrak{q}_d^* \equiv 0 \pmod{\mathfrak{p}}}} r(\beta\lambda, \boldsymbol{\gamma}^{\lambda}, \mathbf{k}) \\
&\quad + \sum_{\substack{k_{d+1}=1 \\ q^m \nmid k_{d+1}}}^{\infty} r(\beta\lambda, \gamma_{d+1}^{\lambda}, k_{d+1}) \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^d \\ \mathrm{tr}_m(\mathfrak{k}) \cdot \mathfrak{q}_d^* \equiv -\mathrm{tr}_m(\mathfrak{k}_{d+1}) \mathfrak{q}_{d+1} \pmod{\mathfrak{p}}}} r(\beta\lambda, \boldsymbol{\gamma}^{\lambda}, \mathbf{k}) \Bigg) \\
&=: \Sigma_1 + \Sigma_2.
\end{aligned} \tag{42}$$

We notice that each summand in Σ_1 is independent of $\mathfrak{q}_{d+1}$ and hence we obtain

$$\begin{aligned}
\Sigma_1 &= \sum_{\substack{k_{d+1}=1 \\ q^m | k_{d+1}}}^{\infty} r(\beta\lambda, \gamma_{d+1}^{\lambda}, k_{d+1}) \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^d \\ \mathrm{tr}_m(\mathfrak{k}) \cdot \mathfrak{q}_d^* \equiv 0 \pmod{\mathfrak{p}}}} r(\beta\lambda, \boldsymbol{\gamma}^{\lambda}, \mathbf{k}) \\
&\overset{(26)}{=} \frac{\mu(\beta\lambda)\gamma_{d+1}^{\lambda}}{q^{\beta\lambda m}} \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^d \\ \mathrm{tr}_m(\mathfrak{k}) \cdot \mathfrak{q}_d^* \equiv 0 \pmod{\mathfrak{p}}}} r(\beta\lambda, \boldsymbol{\gamma}^{\lambda}, \mathbf{k}).
\end{aligned} \tag{43}$$

For the simplification of Σ_2 we observe that $\mathfrak{q}_{d+1} \neq 0$ and $\mathrm{tr}_m(\mathfrak{k}_{d+1}) \neq 0$ as $q^m \nmid k_{d+1}$ in Σ_2. Moreover, $\mathfrak{p}$ is an irreducible polynomial neither dividing $\mathrm{tr}_m(\mathfrak{k}_{d+1})$ nor $\mathfrak{q}_{d+1}$. Thus, $\mathfrak{p}$ does not divide $\mathrm{tr}_m(\mathfrak{k}_{d+1})\mathfrak{q}_{d+1}$. Therefore we have

$$\begin{aligned}
&\sum_{\mathfrak{q}_{d+1} \in G_{q,m}} \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^d \\ \mathrm{tr}_m(\mathfrak{k}) \cdot \mathfrak{q}_d^* \equiv -\mathrm{tr}_m(\mathfrak{k}_{d+1}) \mathfrak{q}_{d+1} \pmod{\mathfrak{p}}}} r(\beta\lambda, \boldsymbol{\gamma}^{\lambda}, \mathbf{k}) \\
&= \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^d \\ \mathrm{tr}_m(\mathfrak{k}) \cdot \mathfrak{q}_d^* \not\equiv 0 \pmod{\mathfrak{p}}}} r(\beta\lambda, \boldsymbol{\gamma}^{\lambda}, \mathbf{k}),
\end{aligned}$$

which leads to the following estimation of Σ_2:

$$
\begin{aligned}
\Sigma_2 &= \frac{1}{q^m-1}\sum_{\substack{k_{d+1}=1\\ q^m\nmid k_{d+1}}}^{\infty} r(\beta\lambda,\gamma_{d+1}^{\lambda},k_{d+1}) \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^d\\ \operatorname{tr}_m(\mathfrak{k})\cdot\mathfrak{q}_d^*\not\equiv 0 \pmod{\mathfrak{p}}}} r(\beta\lambda,\boldsymbol{\gamma}^{\lambda},\mathbf{k})\\
&\leqslant \frac{1}{q^m-1}\Bigg(\sum_{\mathbf{k}\in\mathbb{N}_0^d} r(\beta\lambda,\boldsymbol{\gamma}^{\lambda},\mathbf{k}) - \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^d\\ \operatorname{tr}_m(\mathfrak{k})\cdot\mathfrak{q}_d^*\equiv 0 \pmod{\mathfrak{p}}}} r(\beta\lambda,\boldsymbol{\gamma}^{\lambda},\mathbf{k})\Bigg)\\
&\quad\times\Bigg(\sum_{k_{d+1}=1}^{\infty} r(\beta\lambda,\gamma_{d+1}^{\lambda},k_{d+1})\Bigg)\\
&\overset{\text{Lemma 2.16}}{=} \frac{\mu(\beta\lambda)\gamma_{d+1}^{\lambda}}{q^m-1}\Bigg(\prod_{j=1}^{d}\left(1+\mu(\beta\lambda)\gamma_j^{\lambda}\right) - \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^d\\ \operatorname{tr}_m(\mathfrak{k})\cdot\mathfrak{q}_d^*\equiv 0 \pmod{\mathfrak{p}}}} r(\beta\lambda,\boldsymbol{\gamma}^{\lambda},\mathbf{k})\Bigg).
\end{aligned}
\tag{44}
$$

As an intermediary summary we state what we have shown so far:

$$
\begin{aligned}
\theta(\mathfrak{q}_{d+1}^*) &\overset{(40)}{\leqslant} \Bigg(\frac{1}{q^m-1}\sum_{\mathfrak{q}_{d+1}\in G_{q,m}} \theta^{\lambda}(\mathfrak{q}_{d+1})\Bigg)^{\frac{1}{\lambda}}\\
&\overset{(42)}{\leqslant} (\Sigma_1+\Sigma_2)^{\frac{1}{\lambda}}\\
&\overset{(43),(44)}{\leqslant} \Bigg(\frac{\mu(\beta\lambda)\gamma_{d+1}^{\lambda}}{q^{\beta\lambda m}} \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^d\\ \operatorname{tr}_m(\mathfrak{k})\cdot\mathfrak{q}_d^*\equiv 0 \pmod{\mathfrak{p}}}} r(\beta\lambda,\boldsymbol{\gamma}^{\lambda},\mathbf{k}) + \frac{\mu(\beta\lambda)\gamma_{d+1}^{\lambda}}{q^m-1}\\
&\qquad\times\Big(\prod_{j=1}^{d}\left(1+\mu(\beta\lambda)\gamma_j^{\lambda}\right) - \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^d\\ \operatorname{tr}_m(\mathfrak{k})\cdot\mathfrak{q}_d^*\equiv 0 \pmod{\mathfrak{p}}}} r(\beta\lambda,\boldsymbol{\gamma}^{\lambda},\mathbf{k})\Big)\Bigg)^{\frac{1}{\lambda}}\\
&= \frac{\mu^{\frac{1}{\lambda}}(\beta\lambda)\gamma_{d+1}}{(q^m-1)^{\frac{1}{\lambda}}}\Bigg(\sum_{\substack{\mathbf{k}\in\mathbb{N}_0^d\\ \operatorname{tr}_m(\mathfrak{k})\cdot\mathfrak{q}_d^*\equiv 0 \pmod{\mathfrak{p}}}} r(\beta\lambda,\boldsymbol{\gamma}^{\lambda},\mathbf{k})\Big(\frac{q^m-1}{q^{\beta\lambda m}}-1\Big)\\
&\qquad+\prod_{j=1}^{d}\left(1+\mu(\beta\lambda)\gamma_j^{\lambda}\right)\Bigg)^{\frac{1}{\lambda}}\\
&\leqslant \frac{\mu^{\frac{1}{\lambda}}(\beta\lambda)\gamma_{d+1}}{(q^m-1)^{\frac{1}{\lambda}}}\prod_{j=1}^{d}\left(1+\mu(\beta\lambda)\gamma_j^{\lambda}\right)^{\frac{1}{\lambda}},
\end{aligned}
$$

as $\beta\lambda > 1$.

Using this inequality on (39) and exploiting the hypothesis in (38) finally gives

$$\begin{aligned} e^2_{q^m,d+1}(\mathcal{S}_{\mathfrak{p}}(\mathbf{q}^*_d)) &\overset{(39)}{=} e^2_{q^m,d}(\mathcal{S}_{\mathfrak{p}}(\mathbf{q}^*_d)) + \theta(\mathbf{q}^*_{d+1}) \\ &\overset{(38)}{\leqslant} \frac{1}{(q^m-1)^{\frac{1}{\lambda}}} \prod_{j=1}^{d} \left(1 + \mu(\beta\lambda)\gamma_j^{\lambda}\right)^{\frac{1}{\lambda}} \left(1 + \mu^{\frac{1}{\lambda}}(\beta\lambda)\gamma_{d+1}\right) \\ &\leqslant \frac{1}{(q^m-1)^{\frac{1}{\lambda}}} \prod_{j=1}^{d} \left(1 + \mu(\beta\lambda)\gamma_j^{\lambda}\right)^{\frac{1}{\lambda}} \left(1 + \mu(\beta\lambda)\gamma_{d+1}^{\lambda}\right)^{\frac{1}{\lambda}}, \end{aligned}$$

where, in the last step, we used Jensen's inequality on the last factor. This completes the proof for the $d+1$-case and the result follows by induction for all $d \in \{1, \ldots, s\}$. □

The above theorem allows us to prove various statements concerning the worst-case error for polynomial lattice point sets gained from the CBC construction for which we had only existence results in the general case, i.e. in Section 4.2, or, more precisely, in Theorem 4.15 and Corollaries 4.16 and 4.18.

Corollary 5.10. *Let $\mathfrak{p} \in \mathbb{F}_q[x]$ be an irreducible polynomial with* $\deg(\mathfrak{p}) = m \in \mathbb{N}$. *Furthermore, let $\mathbf{q}^* = (1, \mathfrak{q}^*_2, \ldots, \mathfrak{q}^*_s) \in G^s_{q,m}$ be the s-dimensional vector of polynomials obtained from the component-by-component construction, i.e. Algorithm 5.8. Then, the following assertions are true:*

(i) For all $\delta \in (0, \frac{\beta-1}{2}]$ we have

$$e_{q^m,s}(\mathcal{S}_{\mathfrak{p}}(\mathbf{q}^*)) \leqslant c_{s,\beta,\boldsymbol{\gamma},\delta}\,(q^m)^{-\frac{\beta}{2}+\delta},$$

where

$$c_{s,\beta,\boldsymbol{\gamma},\delta} := 2^{\frac{\beta}{2}-\delta} \prod_{j=1}^{s} \left(1 + \gamma_j^{\frac{1}{\beta-2\delta}}\, \mu\left(\frac{\beta}{\beta-2\delta}\right)\right)^{\frac{\beta}{2}-\delta}.$$

(ii) Under the assumption that

$$\sum_{j=1}^{\infty} \gamma_j^{\frac{1}{\beta-2\delta}} < \infty$$

it follows that

$$c_{s,\beta,\boldsymbol{\gamma},\delta} \leqslant c_{\infty,\beta,\boldsymbol{\gamma},\delta} < \infty$$

and hence

$$e_{q^m,s}(\mathcal{S}_{\mathfrak{p}}(\mathbf{q}^*)) \leqslant c_{\infty,\beta,\boldsymbol{\gamma},\delta}\,(q^m)^{-\frac{\beta}{2}+\delta}\,.$$

(iii) Suppose

$$A := \limsup_{s\to\infty} \frac{\sum_{j=1}^s \gamma_j}{\log s} < \infty.$$

Then, there exists a constant c_η, which is only dependent on $\eta > 0$, such that

$$e_{q^m,s}(\mathcal{S}_{\mathfrak{p}}(\mathbf{q}^*)) \leqslant c_\eta s^{\frac{\mu(\beta)(A+\eta)}{2}} q^{-\frac{m}{2}}$$

for all $\eta > 0$.

(cf. [4, Corollary 4.5])

Proof. As in the proof of [4, Corollary 4.5] we set $\lambda := \frac{1}{\beta-2\delta}$ to find that

$$\frac{1}{\beta} < \underbrace{\frac{1}{\beta-2\delta}}_{=\lambda} \leqslant \frac{1}{\beta-\beta+1} = 1.$$

Thus, we may apply Theorem 5.9 with this value for λ, giving

$$\begin{aligned}
e_{q^m,s}(\mathcal{S}_{\mathfrak{p}}(\mathbf{q}^*)) &\leqslant (q^m-1)^{-\frac{1}{2\lambda}} \prod_{j=1}^s \left(1+\gamma_j^\lambda \mu(\beta\lambda)\right)^{\frac{1}{2\lambda}} \\
&\leqslant \left(\frac{q^m}{2}\right)^{-\frac{1}{2\lambda}} \prod_{j=1}^s \left(1+\gamma_j^\lambda \mu(\beta\lambda)\right)^{\frac{1}{2\lambda}} \\
&= 2^{\frac{\beta}{2}-\delta} \prod_{j=1}^s \left(1+\gamma_j^{\frac{1}{\beta-2\delta}} \mu\left(\frac{\beta}{\beta-2\delta}\right)\right)^{\frac{\beta}{2}-\delta} (q^m)^{-\frac{-\beta}{2}+\delta} \\
&= c_{s,\beta,\boldsymbol{\gamma},\delta}\,(q^m)^{-\frac{\beta}{2}+\delta}
\end{aligned}$$

and hence the first assertion follows.

The proofs of the remaining items are identical to those of Corollaries 4.16 and 4.18 for this special setting of parameters λ and δ and, naturally, one has to refer to item (i) instead of Theorem 4.15. □

From items (ii) and (iii) we learn that, by employing the CBC method, we may exploit strong tractability and tractability of integration in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ respectively.

5.1.2 A Korobov type construction

The second construction method for polynomial lattice point sets which is going to be presented in this thesis is that of a Korobov type. The common feature inherent in such algorithms is that one picks an element of a specific domain (e.g. integers) and forms a vector consisting of successive powers of this element (cf. [5, p. 306]).

For the construction of polynomial lattice point sets we consider an irreducible polynomial $\mathfrak{p} \in \mathbb{F}_q[x]$ of degree $m \in \mathbb{N}$ as well as another polynomial $\mathfrak{q} \in G_{q,m}$ and define the lattice point $\boldsymbol{\mathfrak{q}} = (\mathfrak{q}_1, \ldots, \mathfrak{q}_s)$ by setting

$$\mathfrak{q}_j \equiv \mathfrak{q}^{j-1} \pmod{\mathfrak{p}},$$

where $\deg(\mathfrak{q}_j) < m$ for all $1 \leqslant j \leqslant s$. Most of the time, however, we will adhere to the more convenient notation

$$\boldsymbol{v}_s(\mathfrak{q}) \equiv \left(1, \mathfrak{q}, \mathfrak{q}^2, \ldots, \mathfrak{q}^s\right) \pmod{\mathfrak{p}},$$

(cf. [4, p. 1908]).

Algorithm 5.11 (Korobov type construction)**.** Let all parameters necessary to define the weighted Hilbert space $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ be given and let $s \geqslant 2$. Then:

1. Choose an irreducible polynomial $\mathfrak{p} \in \mathbb{F}_q[x]$ with $\deg(\mathfrak{p}) = m \in \mathbb{N}$.
2. Find $\tilde{\mathfrak{q}} \in G_{q,m}$ which minimizes $e^2_{q^m,s}\left(\mathcal{S}_{\mathfrak{p}}(\boldsymbol{v}_s(\mathfrak{q}))\right)$ over $G_{q,m}$.

(cf. [4, Algorithm 4.6])

Remark 5.12. Similarly to [5, Remarks 10.27 and 10.33] we notice that, in comparison to Algorithm 5.8, we need to try $|G_{q,m}| = q^m - 1$ polynomials for coming up with a point set of the above kind, while this is done in each of the $s - 1$ iteration steps occuring in the CBC construction.

We may estimate the worst-case error for integration in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ using point sets obtained by the above algorithm as follows:

Theorem 5.13. *Let $s \geqslant 2$ and let $\mathfrak{p} \in \mathbb{F}_q[x]$ be irreducible with* $\deg(\mathfrak{p}) = m \in \mathbb{N}$. *For any $\tilde{\mathfrak{q}}$ obtained by Algorithm 5.11 we have*

$$e^2_{q^m,s}(\mathcal{S}_{\mathfrak{p}}(\boldsymbol{v}_s(\tilde{\mathfrak{q}}))) \leqslant \frac{s}{q^m - 1} \prod_{j=1}^{s} \left(1 + \gamma_j \mu_(\beta)\right),$$

where $\mu(\beta)$ is defined in Lemma 2.16.

(cf. [4, Theorem 4.7])

Proof. (Taken from [4, Theorem 4.7])

We begin by defining $M_s(\mathfrak{p})$ as the average of the square worst-case error using a Korobov type polynomial lattice, i.e.

$$M_s(\mathfrak{p}) := \frac{1}{q^m - 1} \sum_{\mathfrak{q} \in G_{q,m}} e^2_{q^m,s}(\mathcal{S}_{\mathfrak{p}}(\boldsymbol{v}_s(\mathfrak{q}))).$$

As $\tilde{\mathfrak{q}}$ is a minimizer of $e^2_{q^m,s}(\mathcal{S}_{\mathfrak{p}}(\boldsymbol{v}_s(\cdot)))$ over $G_{q,m}$ it immediately follows that

$$e^2_{q^m,s}(\mathcal{S}_{\mathfrak{p}}(\boldsymbol{v}_s(\tilde{\mathfrak{q}}))) \leqslant M_s(\mathfrak{p}). \tag{45}$$

Thus, all that remains to be done is to estimate $M_s(\mathfrak{p})$ appropriately. To this end, we simplify as follows

$$\begin{aligned} M_s(\mathfrak{p}) &\overset{\text{Lemma 5.7}}{=} \frac{1}{q^m - 1} \sum_{\mathfrak{q} \in G_{q,m}} \sum_{\mathbf{k} \in \mathcal{D}^*_{\mathfrak{p},\boldsymbol{v}_s(\mathfrak{q})}} r(\beta, \boldsymbol{\gamma}, \mathbf{k}) \\ &= \frac{1}{q^m - 1} \sum_{\mathfrak{q} \in G_{q,m}} \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\} \\ \operatorname{tr}_m(\mathfrak{k}) \cdot \boldsymbol{v}_s(\mathfrak{q}) \equiv 0 \pmod{\mathfrak{p}}}} r(\beta, \boldsymbol{\gamma}, \mathbf{k}) \\ &= \frac{1}{q^m - 1} \sum_{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\}} r(\beta, \boldsymbol{\gamma}, \mathbf{k}) \sum_{\substack{\mathfrak{q} \in G_{q,m} \\ \operatorname{tr}_m(\mathfrak{k}) \cdot \boldsymbol{v}_s(\mathfrak{q}) \equiv 0 \pmod{\mathfrak{p}}}} 1, \end{aligned} \tag{46}$$

leaving us with the task of determining (or estimating) the number of solutions of the congruence

$$\operatorname{tr}_m(\mathfrak{k}_1) + \operatorname{tr}_m(\mathfrak{k}_2)\mathfrak{q} + \cdots + \operatorname{tr}_m(\mathfrak{k}_s)\mathfrak{q}^{s-1} \equiv 0 \pmod{\mathfrak{p}} \tag{47}$$

for each $(k_1, \ldots, k_s) \in \mathbb{N}_0^s \backslash \{\mathbf{0}\}$.

First of all, we consider the case where $\mathbf{k} = q^m \mathbf{l}$ with $\mathbf{l} \in \mathbb{N}_0^s \backslash \{\mathbf{0}\}$. Since $\operatorname{tr}_m(\mathfrak{k}) = \mathbf{0}$, any $\mathfrak{q} \in G_{q,m}$ is a solution to (47). Secondly, if k_1 is not a multiple of q^m and the remaining k_j, $2 \leqslant j \leqslant s$, are, we cannot find any solution at all. In any other case, i.e. $k_j = k_j^* + l_j q^m$ with $0 \leqslant k_j^* < q^m$, $(k_2^*, \ldots, k_s^*) \neq \mathbf{0}$, and $l_j \geqslant 0$ for all $j \in \{2, \ldots, s\}$, there are at most $s - 1$ $\mathfrak{q} \in G_{q,m}$ satisfying (47). To facilitate notation we recall that, for $\mathbf{n} \in \mathbb{N}_0^d$, the maximum value of all of its coordinates is denoted by $\|\mathbf{n}\|_\infty$, where d stands for any finite dimension.

Considering this discussion in (46) yields

$$M_s(\mathfrak{p}) \leqslant \frac{1}{q^m - 1} \sum_{\mathbf{l} \in \mathbb{N}_0^s \backslash \{\mathbf{0}\}} r(\beta, \boldsymbol{\gamma}, q^m \mathbf{l}) \sum_{\mathfrak{q} \in G_{q,m}} 1 + \frac{s-1}{q^m - 1}$$
$$\times \left(\sum_{k_1 = 0}^{\infty} \sum_{(l_2, \ldots, l_s) \in \mathbb{N}_0^{s-1}} \sum_{\substack{(k_2^*, \ldots, k_s^*) \in \mathbb{N}_0^{s-1} \backslash \{\mathbf{0}\} \\ \|(k_2^*, \ldots, k_s^*)\|_\infty < q^m}} r(\beta, \gamma_1, k_1) \prod_{j=2}^{s} r(\beta, \gamma_j, k_j^* + q^m l_j) \right).$$

The identities

$$\sum_{\mathbf{l} \in \mathbb{N}_0^s} r(\beta, \boldsymbol{\gamma}, q^m \mathbf{l}) = \prod_{j=1}^{s} \left(1 + \gamma_j \frac{\mu(\beta)}{q^{m\beta}} \right)$$

and

$$\sum_{\mathbf{l} \in \mathbb{N}_0^{s-1}} \sum_{\substack{\mathbf{k}^* \in \mathbb{N}_0^{s-1} \backslash \{\mathbf{0}\} \\ \|\mathbf{k}^*\|_\infty < q^m}} r(\beta, \boldsymbol{\gamma}, \mathbf{k}^* + q^m \mathbf{l}) = \prod_{j=1}^{s-1} (1 + \gamma_j \mu(\beta)) - \prod_{j=1}^{s-1} \left(1 + \gamma_j \frac{\mu(\beta)}{q^{m\beta}} \right)$$

have already occurred in Lemma 4.14 in (26) and (27). Applying this to the respective terms and Lemma 2.16 to the sum over k_1 allows us to simplify further

$$M_s(\mathfrak{p}) \leqslant \prod_{j=1}^{s} \left(1 + \gamma_j \frac{\mu(\beta)}{q^{m\beta}} \right) - 1 + \frac{s-1}{q^m - 1} (1 + \gamma_1 \mu(\beta))$$
$$\times \left(\prod_{j=2}^{s} (1 + \gamma_j \mu(\beta)) - \prod_{j=2}^{s} \left(1 + \gamma_j \frac{\mu(\beta)}{q^{m\beta}} \right) \right).$$

Once again, we need to cite Lemma 4.14, but now Equation (28), in which one can see that

$$\prod_{j=1}^{s} \left(1 + \gamma_j \frac{\mu(\beta)}{q^{m\beta}} \right) - 1 \leqslant \frac{1}{q^m} \prod_{j=1}^{s} (1 + \gamma_j \mu(\beta)).$$

Thus, we finally arrive at

$$\begin{aligned} M_s(\mathfrak{p}) &\leqslant \frac{1}{q^m}\prod_{j=1}^{s}(1+\gamma_j\mu(\beta)) + \frac{s-1}{q^m-1}(1+\gamma_1\mu(\beta))\prod_{j=2}^{s}(1+\gamma_j\mu(\beta)) \\ &\leqslant \frac{s}{q^m-1}\prod_{j=1}^{s}(1+\gamma_j\mu(\beta)). \end{aligned}$$

Inserting this into the inequality given in (45) completes the proof. □

We immediately notice that, compared to worst-case error for point sets gained from the CBC construction method, the dimension s appears as a stand-alone factor in the above theorem. Thus, we are only able to present a reduced version of Corollary 5.10.

Corollary 5.14. *Let $\mathfrak{p}$ be an irreducible polynomial over $\mathbb{F}_q$ with $\deg(\mathfrak{p}) = m \in \mathbb{N}$ and let $\tilde{\mathfrak{q}} \in G_{q,m}$ be the polynomial obtained from Algorithm 5.11. Then, the following assertions are true:*

(i) For all $\delta \in \left(0, \frac{\beta-1}{2}\right]$ we have

$$e_{q^m,s}(\mathcal{S}_{\mathfrak{p}}(\boldsymbol{v}_s(\tilde{\mathfrak{q}}))) \leqslant c_{s,\beta,\boldsymbol{\gamma},\delta}\, s^{\frac{\beta}{2}-\delta}\left(q^m\right)^{-\frac{\beta}{2}+\delta},$$

where

$$c_{s,\beta,\boldsymbol{\gamma},\delta} := 2^{\frac{\beta}{2}-\delta}\prod_{j=1}^{s}\left(1+\gamma_j^{\frac{1}{\beta-2\delta}}\mu\left(\frac{\beta}{\beta-2\delta}\right)\right)^{\frac{\beta}{2}-\delta}.$$

(ii) Suppose

$$A := \limsup_{s\to\infty}\frac{\sum_{j=1}^{s}\gamma_j}{\log s} < \infty.$$

Then, there exists a constant c_η which solely dependens on $\eta > 0$, such that

$$e_{q^m,s}(\mathcal{S}_{\mathfrak{p}}(\boldsymbol{v}_s(\tilde{\mathfrak{q}}))) \leqslant c_\eta s^{\frac{1+\mu(\beta)(A+\eta)}{2}} q^{-\frac{m}{2}}.$$

Hence, the worst-case error depends at most polynomially on the dimension s.

(cf. [4, Corollary 4.8])

Proof. (Taken from [4, Corollary 4.8])

(i) We consider the dependency of the worst-case error on the parameters β and $\boldsymbol{\gamma}$ by writing $e_{q^m,s}(\beta, \boldsymbol{\gamma}, \mathcal{S}_{\mathfrak{p}}(\boldsymbol{v}_s(\mathfrak{q})))$. Furthermore, we define $\lambda := 1/(\beta - 2\delta)$. Note that $1/\beta < \lambda \leqslant 1$. Using Lemma 5.7 and applying Jensen's inequality gives

$$\begin{aligned} e^2_{q^m,s}(\beta, \boldsymbol{\gamma}, \mathcal{S}_{\mathfrak{p}}(\boldsymbol{v}_s(\mathfrak{q}))) &\leqslant \sum_{\mathbf{k} \in \mathcal{D}^*_{\mathfrak{p}, \boldsymbol{v}_s(\mathfrak{q})}} r(\beta, \boldsymbol{\gamma}, \mathbf{k}) \\ &\leqslant \left(\sum_{\mathbf{k} \in \mathcal{D}^*_{\mathfrak{p}, \boldsymbol{v}_s(\mathfrak{q})}} r(\beta\lambda, \boldsymbol{\gamma}^\lambda, \mathbf{k}) \right)^{\frac{1}{\lambda}} \\ &= \left(e^2_{q^m,s} \left(\beta\lambda, \boldsymbol{\gamma}^\lambda, \mathcal{S}_{\mathfrak{p}}(\boldsymbol{v}_s(\mathfrak{q}))\right) \right)^{\frac{1}{\lambda}}. \end{aligned}$$

Theorem 5.13 now implies that there exists a $\mathfrak{q}_* \in G_{q,m}$ such that

$$\begin{aligned} e^2_{q^m,s} \left(\beta\lambda, \boldsymbol{\gamma}^\lambda, \mathcal{S}_{\mathfrak{p}}(\boldsymbol{v}_s(\mathfrak{q}_*))\right) &\leqslant \frac{s}{q^m - 1} \prod_{j=1}^{s} \left(1 + \gamma_j^\lambda \mu(\beta\lambda)\right) \\ &\leqslant \frac{2s}{q^m} \prod_{j=1}^{s} \left(1 + \gamma_j^\lambda \mu(\beta\lambda)\right). \end{aligned}$$

Since $\tilde{\mathfrak{q}}$ is a minimizer of the square worst-case error, we obtain

$$\begin{aligned} e_{q^m,s} \left(\beta, \boldsymbol{\gamma}, \mathcal{S}_{\mathfrak{p}}(\boldsymbol{v}_s(\tilde{\mathfrak{q}}))\right) &\leqslant \left(e^2 \left(\beta\lambda, \boldsymbol{\gamma}^\lambda, \mathcal{S}_{\mathfrak{p}}(\boldsymbol{v}_s(\tilde{\mathfrak{q}}))\right) \right)^{\frac{1}{2\lambda}} \\ &= \left(\frac{2s}{q^m}\right)^{\frac{\beta}{2} - \delta} \prod_{j=1}^{s} \left(1 + \gamma_j^{\frac{1}{\beta - 2\delta}} \mu\left(\frac{\beta}{\beta - 2\delta}\right)\right)^{\frac{\beta}{2} - \delta}. \end{aligned}$$

(ii) The proof of this result follows exactly the same pattern as that of Corollary 4.18 and will therefore be omitted.

□

Finally, we compare the results for the two construction methods which were introduced in this section. Although the Korobov type construction takes less time to come up with a polynomial lattice (see Remark 5.12), we learn from Corollary 5.14 that, in this case, the worst-case error differs by a positive power of s, compared to that obtained by the CBC construction. More importantly, however, this factor deprives us of the possibility to exploit strong tractability if $\sum_{j=1}^{\infty} \gamma_j^\lambda < \infty$. So, in principle, we may say that Algorithm 5.8 yields better results than Algorithm 5.11, (cf. [5, Remark 10.33] and [4, p. 1911]).

5.2 The construction method by Niederreiter

This method, too, uses formal Laurent series for the determination of the generating matrices. In this approach, however, one uses different irreducible polynomials as denominators instead of concentrating on the variation of the numerator, compared to the construction of polynomial lattice point sets. The following construction scheme was introduced by H. Niederreiter in [14] and can also be found in [5, p. 264].

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s \in \mathbb{F}_q[x]$ be monic, irreducible and mutually distinct with $\deg(\mathfrak{p}_j) = e_j \in \mathbb{N}$ for all $1 \leqslant j \leqslant s$. Furthermore, for $1 \leqslant j \leqslant s$ and all $1 \leqslant i \leqslant m$ we consider

$$\frac{x^k}{\mathfrak{p}_j(x)^i} = \sum_{r=0}^{\infty} a^{(j)}(i,k,r) x^{-r-1}$$

for all integers $0 \leqslant k < e_j$ and set

$$c_{i,r}^{(j)} = a^{(j)}(Q+1,k,r) \in \mathbb{F}_q,$$

where $0 \leqslant r \leqslant m-1$ and the integers Q and $0 \leqslant k < e_j$ are chosen such that

$$i - 1 = Qe_j + k$$

holds. Subsequently, we collect the coefficients $c_{i,r}^{(j)}$ to form the generating matrices

$$C_j = \left(c_{i,r}^{(j)} \right)_{\substack{i=1,\ldots,m \\ r=0,\ldots,m-1}},$$

$1 \leqslant j \leqslant s$.

Before we are able to proceed further we will have to work ourselves through a series of lemmas, which aim at showing another bound for the worst-case error. As a reward, this will vastly facilitate the estimation of the worst-case error employing digital nets of the above kind.

Lemma 5.15. *Let $\mathcal{P}$ be a digital (t,m,s)-net over $\mathbb{F}_q$ whose generating matrices $C_1, \ldots, C_s$ are non-singular. Furthermore, we define*

$$[d] := \{1, \ldots, d\}, \ d \in \mathbb{N},$$

and

$$\mathcal{D}_{q^m} := \left\{ \mathbf{k} \in \mathbb{N}_0^s\colon \ \|\mathbf{k}\|_\infty < q^m \text{ and } \sum_{j=1}^{s} C_j^\top \varphi(k_j) = \mathbf{0} \right\}$$

and set $\mathcal{D}^*_{q^m} := \mathcal{D}_{q^m} \backslash \{\mathbf{0}\}$.

Then we have

$$\sum_{\mathrm{u} \subsetneq [s]} \left(\sum_{\mathbf{k} \in \mathcal{D}^*_{q^m}} \prod_{j \in \mathrm{u}} r(\beta, \gamma_j, k_j) \right) \left(\prod_{j \in [s] \backslash \mathrm{u}} \gamma_j \frac{\mu(\beta)}{q^{m\beta}} \right)$$
$$\leqslant \frac{1}{q^{m\beta}} \prod_{j=1}^{s} (1 + 2\gamma_j \mu(\beta)) - \prod_{j=1}^{s} \left(1 + \gamma_j \frac{\mu(\beta)}{q^{m\beta}} \right) + 1.$$

(cf. [3, p. 286])

Proof. (Adapted from [3, pp. 285f.]).

Let $\mathrm{u} = \{u_1, \ldots, u_e\}$ be a proper subset of $[s]$, i.e. $e < s$, and consider $k_{u_1}, \ldots, k_{u_e} \in \{0, \ldots, q^m - 1\}$ fixed. Moreover, let $\{j_1, \ldots, j_{s-|\mathrm{u}|}\} := [s] \backslash \mathrm{u}$.

Thus, the condition $\mathbf{k} \in \mathcal{D}_{q^m}$ is equivalent to

$$C_{j_1}^\top \varphi(k_{j_1}) + \cdots + C_{j_{s-|\mathrm{u}|}}^\top \varphi(k_{j_{s-|\mathrm{u}|}}) = \mathbf{b},$$

where $\mathbf{b} = -C_{u_1}^\top \varphi(k_{u_1}) - \cdots - C_{u_e}^\top \varphi(k_{u_e}) \in \mathbb{F}_q^m$. For an appropriate vector $\mathbf{d} = \mathbf{d}(k_{j_1}, \ldots, k_{j_{s-|\mathrm{u}|-1}})$ we may write

$$C_{j_{s-|\mathrm{u}|}}^\top \varphi(k_{j_{s-|\mathrm{u}|}}) = \mathbf{d}.$$

The latter equation admits of $q^{m(s-|\mathrm{u}|-1)}$ right handsides, for each of which there exists exactly one solution, since $C_{j_{s-|\mathrm{u}|}}$ is regular by assumption. Thus, for fixed $k_{u_1}, \ldots, k_{u_e}$, the maximum number of $\mathbf{k} \in \mathcal{D}_{q^m}$ is bounded by $q^{m(s-|\mathrm{u}|-1)}$ and hence

$$\begin{aligned} \sum_{\mathbf{k} \in \mathcal{D}^*_{q^m}} \prod_{j \in \mathrm{u}} r(\beta, \gamma_j, k_j) \quad &\leqslant \quad q^{m(s-|\mathrm{u}|-1)} \prod_{j \in \mathrm{u}} \left(\sum_{k=0}^{q^m-1} r(\beta, \gamma_j, k) \right) - 1 \\ &\overset{\text{Lemma 2.16}}{\leqslant} \quad q^{m(s-|\mathrm{u}|-1)} \prod_{j \in \mathrm{u}} (1 + \gamma_j \mu(\beta)) - 1. \end{aligned}$$

Note that we needed to substract 1, as we have allowed $\mathbf{k}$ to be the zero vector in the above discussion. Using this inequality and the fact that $\beta > 1$

we obtain

$$\begin{aligned}
&\sum_{\mathrm{u}\subsetneq[s]}\left(\sum_{(k_1,\ldots,k_s)\in\mathcal{D}^*_{q^m}}\prod_{j\in\mathrm{u}} r(\beta,\gamma_j,k_j)\right)\left(\prod_{j\in[s]\setminus\mathrm{u}}\gamma_j\frac{\mu(\beta)}{q^{m\beta}}\right)\\
&\leqslant \sum_{\mathrm{u}\subsetneq[s]}\left(q^{m\beta(s-|\mathrm{u}|-1)}\prod_{j\in\mathrm{u}}(1+\gamma_j\mu(\beta))-1\right)\left(\prod_{j\in[s]\setminus\mathrm{u}}\gamma_j\frac{\mu(\beta)}{q^{m\beta}}\right)\\
&=\sum_{\mathrm{u}\subsetneq[s]} q^{m\beta(s-|\mathrm{u}|-1)}\prod_{j\in\mathrm{u}}(1+\gamma_j\mu(\beta))\prod_{j\in[s]\setminus\mathrm{u}}\gamma_j\frac{\mu(\beta)}{q^{m\beta}}-\sum_{\mathrm{u}\subseteq[s]}\prod_{j\in[s]\setminus\mathrm{u}}\gamma_j\frac{\mu(\beta)}{q^{m\beta}}+1\\
&\leqslant\frac{1}{q^{m\beta}}\sum_{\mathrm{u}\subseteq[s]}\prod_{j\in\mathrm{u}}(1+\gamma_j\mu(\beta))\prod_{j\in[s]\setminus\mathrm{u}}\gamma_j\mu(\beta)-\prod_{j=1}^{s}\left(1+\gamma_j\frac{\mu(\beta)}{q^{m\beta}}\right)+1\\
&=\frac{1}{q^{m\beta}}\prod_{j=1}^{s}(1+2\gamma_j\mu(\beta))-\prod_{j=1}^{s}\left(1+\gamma_j\frac{\mu(\beta)}{q^{m\beta}}\right)+1.
\end{aligned}$$

□

The proof of the next lemma will be omitted, as its result is more of a technical nature and does hardly provide any valuable information concerning digital nets.

Lemma 5.16. *Let $b > 1$ be a real number and let $k,\ t_0 \in \mathbb{N}$. Then the following inequality holds*

$$\sum_{t=t_0}^{\infty}\binom{t+k-1}{k-1}b^{-t}\leqslant b^{-t_0}\binom{t_0+k-1}{k-1}\left(1-\frac{1}{b}\right)^{-k}.$$

Proof. See [7, Lemma 6]. □

This result comes in handy for the proof of the following lemma.

Lemma 5.17. *Let $\mathrm{u} = \{u_1,\ldots,u_e\}$ be a non-empty subset of $[s]$ and let $\mathcal{P}$ be a digital (t,m,s)-net over $\mathbb{F}_q$ with generating matrices $C_1,\ldots,C_s$. Furthermore, assume that the projection of $\mathcal{P}$ onto the coordinates in u constitutes a digital $(t_\mathrm{u},m,|u|)$-net, for some $t_\mathrm{u}\leqslant m$. Then*

$$\mathcal{B}(\mathrm{u})\leqslant\left(\frac{q-1}{q^{\beta-1}-1}\right)^{|\mathrm{u}|}\frac{2(m-t_\mathrm{u}+2)^{|\mathrm{u}|-1}}{q^{\beta(m-t_\mathrm{u}+1-2|\mathrm{u}|)}}\prod_{j\in\mathrm{u}}\gamma_j,$$

where $\mathcal{B}(\mathrm{u})$ is defined by

$$\mathcal{B}(\mathrm{u}) := \sum_{\substack{k_{u_1},\ldots,k_{u_e}=0 \\ C_{u_1}^\top \varphi(k_{u_1})+\cdots+C_{u_e}^\top \varphi(k_{u_e})=\mathbf{0}}}^{q^m-1} \prod_{j\in\mathrm{u}} r(\beta,\gamma_j,k_j).$$

(cf. [3, p. 287])

Proof. Without loss of generality we choose $\mathrm{u} = [e]$ and obtain

$$\mathcal{B}(\mathrm{u}) = \sum_{a_1,\ldots,a_e=0}^{m-1} q^{-\beta(a_1+\cdots+a_e)} \prod_{j=1}^{e} \gamma_j \underbrace{\sum_{k_1=q^{a_1}}^{q^{a_1+1}-1} \cdots \sum_{k_e=q^{a_e}}^{q^{a_e+1}-1}}_{C_1^\top \varphi(k_1)+\cdots+C_e^\top \varphi(k_e)=\mathbf{0}} 1, \tag{48}$$

as we have already done earlier in the proof of Lemma 2.16 or as can be seen similarly in [3, p. 286].

The remaining part considers the proof of [2, Lemma 7], where a different result is shown, but the main ideas relate, nevertheless. Next, we aim at reformulating the condition $\sum_{j=1}^{e} C_j^\top \varphi(k_j) = \mathbf{0}$. To this end, let $\mathbf{c}_{j,i}^\top$ be the ith row vector of the generating matrix C_j, $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant s$. Additionally, we denote the lth q-adic digit of k_j by $\kappa_{j,l}$. Thus, the above-mentioned condition is equivalent to

$$\begin{aligned}\mathbf{0} = {} & \mathbf{c}_{1,1}\varphi_1(\kappa_{1,0}) + \cdots + \mathbf{c}_{1,a_1}\varphi_1(\kappa_{1,a_1-1}) + \mathbf{c}_{1,a_1+1}\varphi_1(\kappa_{1,a_1}) \\ & + \\ & \mathbf{c}_{2,1}\varphi_1(\kappa_{2,0}) + \cdots + \mathbf{c}_{2,a_2}\varphi_1(\kappa_{2,a_2-1}) + \mathbf{c}_{2,a_2+1}\varphi_1(\kappa_{2,a_2}) \\ & + \\ & \vdots \\ & + \\ & \mathbf{c}_{e,1}\varphi_1(\kappa_{e,0}) + \cdots + \mathbf{c}_{e,a_e}\varphi_1(\kappa_{e,a_e-1}) + \mathbf{c}_{e,a_e+1}\varphi_1(\kappa_{e,a_e}).\end{aligned} \tag{49}$$

Since, by assumption, the projection of the digital net $\mathcal{P}$ onto the coordinates in u is a digital $(t_{\mathrm{u}}, m, |\mathrm{u}|)$-net, it follows from Lemma 3.5 that the vectors

$$\mathbf{c}_{1,1}, \ldots, \mathbf{c}_{1,a_1+1}, \mathbf{c}_{2,1}, \ldots, \mathbf{c}_{e,1}, \ldots, \mathbf{c}_{e,a_e+1}$$

are linearly independent, provided that

$$\sum_{i=1}^{e}(a_i+1) \leqslant m - t_{\mathrm{u}},$$

which implies that, with this restraint put on $a_1, \ldots, a_e$, a non-zero solution $k_1, \ldots, k_e$ of (49) cannot exist. Consequently, we only need to investigate the case where

$$\sum_{i=1}^{e} a_i \geqslant m - t_{\mathrm{u}} - e + 1.$$

For this reason, let

$$\begin{aligned} A &:= (\mathbf{c}_{1,1}, \ldots, \mathbf{c}_{1,a_1}, \mathbf{c}_{2,1}, \ldots, \mathbf{c}_{e,1}, \ldots, \mathbf{c}_{e,a_e}) && \in \mathbb{F}_q^{m\times(a_1+\cdots+a_e)}, \\ \mathbf{f}_{\kappa_{1,a_1},\ldots,\kappa_{e,a_e}} &:= -\big(\mathbf{c}_{1,a_1+1}\varphi_1(\kappa_{1,a_1}) + \mathbf{c}_{2,a_2+1}\varphi_1(\kappa_{2,a_2}) + \cdots \\ &\quad + \cdots + \mathbf{c}_{e,a_e+1}\varphi_1(\kappa_{e,a_e})\big) && \in \mathbb{F}_q^{m} \end{aligned}$$

and

$$\begin{aligned} \mathbf{k} &:= \big(\varphi_1(\kappa_{1,0}), \ldots, \varphi_1(\kappa_{1,a_1-1}), \varphi_1(\kappa_{2,0}), \ldots \\ &\quad \ldots, \varphi_1(\kappa_{e,0}), \ldots, \varphi_1(\kappa_{e,a_e-1})\big)^{\top} && \in \mathbb{F}_q^{a_1+\cdots+a_e}. \end{aligned}$$

Now we can rewrite (49) in the following way:

$$A\mathbf{k} = \mathbf{f}_{\kappa_{1,a_1},\ldots,\kappa_{e,a_e}}. \tag{50}$$

From Lemma 3.5 we know that for the rank of the matrix A, we denote it by $\mathrm{rank}(A)$, we have

$$\mathrm{rank}(A) = a_1 + \cdots + a_e, \quad \text{if } a_1 + \cdots + a_e \leqslant m - t_{\mathrm{u}}$$

and

$$\mathrm{rank}(A) \geqslant m - t_{\mathrm{u}}$$

otherwise.

Thus, if we denote the space of solutions of the linear system of equations $A\mathbf{k} = \mathbf{0}$ by L, it follows that

$$\dim(L) = 0, \quad \text{if } a_1 + \cdots + a_e \leqslant m - t_{\mathrm{u}}$$

and

$$\dim(L) \leqslant a_1 + \cdots + a_e - m + t_{\mathrm{u}}$$

in the other case. Hence,

$$\#\left\{\mathbf{k} \in \mathbb{F}_q^{a_1+\dots+a_e} : \; A\mathbf{k} = \mathbf{f}_{\kappa_{1,a_1},\dots,\kappa_{e,a_e}}\right\} \leqslant \begin{cases} 1 & \text{if } \sum_{j=1}^{e} a_j \leqslant m - t_{\mathrm{u}}, \\ q^{\sum_{j=1}^{e} a_j - m + t_{\mathrm{u}}} & \text{else.} \end{cases}$$

This finally allows us to determine an upper bound for $\mathcal{B}(\mathrm{u})$, based on Equation (48):

$$\begin{aligned}
\mathcal{B}(\mathrm{u}) &= \sum_{a_1,\dots,a_e=0}^{m-1} q^{-\beta(a_1+\dots+a_e)} \prod_{j=1}^{e} \gamma_j \underbrace{\sum_{k_1=q^{a_1}}^{q^{a_1+1}-1} \cdots \sum_{k_e=q^{a_e}}^{q^{a_e+1}-1}}_{C_1^\top \varphi(k_1)+\dots+C_e^\top \varphi(k_e)=\mathbf{0}} 1 \\
&= \sum_{a_1,\dots,a_e=0}^{m-1} q^{-\beta(a_1+\dots+a_e)} \prod_{j=1}^{e} \gamma_j \sum_{\kappa_{1,a_1},\dots,\kappa_{e,a_e}=1}^{q-1} \sum_{\substack{\mathbf{k}\in\mathbb{F}_q^{a_1+\dots+a_e} \\ A\mathbf{k}=f_{\kappa_{1,a_1},\dots,\kappa_{e,a_e}}}} 1 \\
&\leqslant \sum_{\substack{a_1,\dots,a_e=0 \\ a_1+\dots+a_e \geqslant m-t_{\mathrm{u}}-e+1}}^{m-1} q^{-\beta(a_1+\dots+a_e)} \prod_{j=1}^{e} \gamma_j \sum_{\kappa_{1,a_1},\dots,\kappa_{e,a_e}=1}^{q-1} 1 \\
&\quad \times \begin{cases} 1 & \text{if } \sum_{l=1}^{e} a_l \leqslant m - t_{\mathrm{u}}, \\ q^{\sum_{l=1}^{e} a_l - m + t_{\mathrm{u}}} & \text{if } \sum_{l=1}^{e} a_l > m - t_{\mathrm{u}} \end{cases} \\
&= (q-1)^e \sum_{\substack{a_1,\dots,a_e=0 \\ m-t_{\mathrm{u}}-e+1 \leqslant a_1+\dots+a_e \leqslant m-t_{\mathrm{u}}}}^{m-1} q^{-\beta(a_1+\dots+a_e)} \prod_{j=1}^{e} \gamma_j \\
&\quad +(q-1)^e \sum_{\substack{a_1,\dots,a_e=0 \\ a_1+\dots+a_e > m-t_{\mathrm{u}}}}^{m-1} q^{(1-\beta)(a_1+\dots+a_e)-m+t_{\mathrm{u}}} \prod_{j=1}^{e} \gamma_j \\
&=: \left((q-1)^e \prod_{j=1}^{e} \gamma_j\right)\left(\Sigma_1 + \Sigma_2\right).
\end{aligned} \tag{51}$$

We can rewrite Σ_2 as follows:

$$\begin{aligned}
\Sigma_2 &= q^{t_{\mathrm{u}}-m} \sum_{\substack{a_1,\dots,a_e=0 \\ a_1+\dots+a_e > m-t_{\mathrm{u}}}}^{m-1} q^{-(\beta-1)(a_1+\dots+a_e)} \\
&= q^{t_{\mathrm{u}}-m} \sum_{l=m-t_{\mathrm{u}}+1}^{e(m-1)} q^{-l(\beta-1)} \sum_{\substack{a_1,\dots,a_e=0 \\ a_1+\dots+a_e=l}}^{m-1} 1.
\end{aligned}$$

Since the number of non-negative integer solutions $(a_1, \dots, a_e)$ to the equation $a_1 + \cdots + a_e = l$ is given by $\binom{l+e-1}{e-1}$, we obtain

$$\begin{aligned}
\Sigma_2 &\leqslant q^{t_\mathrm{u}-m} \sum_{l=m-t_\mathrm{u}+1}^{\infty} \binom{l+e-1}{e-1} q^{-l(\beta-1)} \\
&\overset{\text{Lemma 5.16}}{\leqslant} q^{t_\mathrm{u}-m} q^{-(\beta-1)(m-t_\mathrm{u}+1)} \binom{m-t_\mathrm{u}+e}{e-1} \left(\frac{q^{\beta-1}-1}{q^{\beta-1}}\right)^{-e} \\
&= q^{-\beta(m-t_\mathrm{u}+1)+(\beta-1)e+1} \left(q^{\beta-1}-1\right)^{-e} \binom{m-t_\mathrm{u}+e}{e-1} \\
&\leqslant q^{-\beta(m-t_\mathrm{u}+1-2e)} \left(q^{\beta-1}-1\right)^{-e} \binom{m-t_\mathrm{u}+e}{e-1},
\end{aligned}$$

and since for $e \geqslant 2$ we have (note that the inequality below is trivially fulfilled for $e = 1$)

$$\binom{m-t_\mathrm{u}+e}{e-1} = \frac{m-t_\mathrm{u}+2}{1} \cdot \frac{m-t_\mathrm{u}+3}{2} \cdots \frac{m-t_\mathrm{u}+e}{e-1} \leqslant (m-t_\mathrm{u}+2)^{e-1},$$

we finally arrive at

$$\Sigma_2 \leqslant q^{-\beta(m-t_\mathrm{u}+1-2e)} \left(q^{\beta-1}-1\right)^{-e} (m-t_\mathrm{u}+2)^{e-1}. \tag{52}$$

Now we turn to the estimation of Σ_1 as given in (51), i.e.

$$\Sigma_1 = \sum_{\substack{a_1,\ldots,a_e=0 \\ m-t_\mathrm{u}-e+1 \leqslant a_1+\cdots+a_e \leqslant m-t_\mathrm{u}}}^{m-1} q^{-\beta(a_1+\cdots+a_e)}.$$

In order to determine an upper bound for this expression, we need to distinguish between two cases, beginning with that where $m-t_\mathrm{u} > e-1$. Hence, by employing the same quantitative arguments concerning non-negative integer solutions to $a_1 + \cdots + a_e = l$ as we have used above, we get

$$\begin{aligned}
\Sigma_1 &= \sum_{l=m-t_\mathrm{u}-e+1}^{m-t_\mathrm{u}} q^{-\beta l} \sum_{\substack{a_1,\ldots,a_e=0 \\ a_1+\cdots+a_e=l}}^{m-1} 1 \\
&\leqslant \sum_{l=m-t_\mathrm{u}-e+1}^{\infty} \binom{l+e-1}{e-1} q^{-\beta l} \\
&\overset{\text{Lemma 5.16}}{\leqslant} q^{-\beta(m-t_\mathrm{u}-e+1)} \binom{m-t_\mathrm{u}}{e-1} \left(\frac{q^{\beta}}{q^{\beta}-1}\right)^{e} \\
&\leqslant q^{-\beta(m-t_\mathrm{u}-2e+1)} \left(q^{\beta-1}-1\right)^{-e} \binom{m-t_\mathrm{u}+e}{e-1} \\
&\leqslant q^{-\beta(m-t_\mathrm{u}-2e+1)} \left(q^{\beta-1}-1\right)^{-e} (m-t_\mathrm{u}+2)^{e-1},
\end{aligned}$$

where the last inequality has already been shown in the estimation of Σ_2.

If $m - t_{\mathrm{u}} \leqslant e - 1$ we can do the following, based on some facts previously used within this proof:

$$\begin{aligned}
\Sigma_1 \quad &\leqslant \quad \sum_{l=0}^{\infty} q^{-\beta l} \binom{l+e-1}{e-1} \\
&\overset{\text{Lemma 5.16}}{\leqslant} \quad 1 + q^{-\beta} \binom{e}{e-1} \left(\frac{q^{\beta}}{q^{\beta}-1} \right)^{e} \\
&= \quad \frac{\left(q^{\beta}-1\right)^{e} + q^{\beta(e-1)} e}{\left(q^{\beta}-1\right)^{e}} \\
&\leqslant \quad \frac{q^{\beta e}\left(1 + e q^{-\beta}\right)}{\left(q^{\beta-1}-1\right)^{e}}. \qquad (53)
\end{aligned}$$

It is rather obvious that

$$1 + e q^{-\beta} \leqslant \left(m - t_{\mathrm{u}} + 2\right)^{e-1} \qquad (54)$$

holds for $e = 2$. Now, suppose the above holds for some $2 \leqslant e < s$. Then we obtain

$$\begin{aligned}
1 + (e+1) q^{-\beta} &\leqslant \left(m - t_{\mathrm{u}} + 2\right)^{e-1} + q^{-\beta} \\
&\leqslant \left(m - t_{\mathrm{u}} + 2\right)^{e-1} + 1 \\
&\leqslant 2\left(m - t_{\mathrm{u}} + 2\right)^{e-1} \\
&\leqslant \left(m - t_{\mathrm{u}} + 2\right)^{e}
\end{aligned}$$

and therefore (54) holds for all $2 \leqslant e \leqslant s$. Together with the fact that $e - 1 \geqslant m - t_{\mathrm{u}}$ implies $e \leqslant -m + t_u - 1 + 2e$, it follows from (53) that

$$\Sigma_1 \leqslant q^{-\beta(m - t_{\mathrm{u}} - 2e + 1)} \left(q^{\beta-1} - 1\right)^{-e} (m - t_{\mathrm{u}} + 2)^{e-1}, \qquad (55)$$

as we have also obtained for $e - 1 < m - t_{\mathrm{u}}$. For $e = 1$ this result can be easily derived from the definition of Σ_1.

Inserting (54) and (52) into (51) finally gives

$$\begin{aligned}
\mathcal{B}(\mathrm{u}) &\leqslant \left((q-1)^{e} \prod_{j=1}^{e} \gamma_j\right) \left(\Sigma_1 + \Sigma_2\right) \\
&\leqslant \left((q-1)^{e} \prod_{j=1}^{e} \gamma_j\right) \left(2 q^{-\beta(m - t_{\mathrm{u}} + 1 - 2e)} \left(q^{\beta-1} - 1\right)^{-e} \left(m - t_{\mathrm{u}} + 2\right)^{e-1}\right)
\end{aligned}$$

and since $e = |\mathrm{u}|$ the result follows. □

We are now in a position to find an upper bound for the worst-case error for digital nets with regular generating matrices in terms of the quality parameters of their projections, which were previously denoted by t_{u}.

Lemma 5.18. *Let $\mathcal{P}$ be a digital (t, m, s)-net over $\mathbb{F}_q$ generated by the regular matrices $C_1, \ldots, C_s \in \mathbb{F}_q^{m\times m}$. Under the assumption that for each non-empty $\mathrm{u} \subseteq \{1, \ldots, s\}$ the projection of $\mathcal{P}$ on the coordinates in u constitutes a $(t_{\mathrm{u}}, m, |\mathrm{u}|)$-net for some integer $0 \leqslant t_{\mathrm{u}} \leqslant m$, the worst-case error for integration in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ is bounded by*

$$e^2_{q^m,s}(\mathcal{P}) \leqslant \frac{1}{q^{\beta m}} \left(1 + \sum_{\varnothing \neq \mathrm{u} \subseteq \{1,\ldots,s\}} q^{\beta t_{\mathrm{u}}} \prod_{j \in \mathrm{u}} \left(q^{\beta+1}(m+2)\mu(\beta)\gamma_j \right) \right).$$

(cf. [3, Lemma 4])

Proof. The most important steps of this proof can also be found in the proof of [3, Lemma 4].

Once again, we use Theorem 4.12 to find that

$$e^2_{q^m,s}(\mathcal{P}) = \sum_{\substack{\mathbf{k}=(k_1,\ldots,k_s)\in\mathbb{N}_0^s\setminus\{\mathbf{0}\} \\ C_1^\top\varphi(k_1)+\cdots+C_s^\top\varphi(k_s)=\mathbf{0}}} r(\beta, \boldsymbol{\gamma}, \mathbf{k})$$

and we notice that, if $\mathbf{k}$ is a multiple of q^m, i.e. there exists an $\mathbf{l} \in \mathbb{N}_0^s\setminus\{\mathbf{0}\}$ such that $\mathbf{k} = q^m\mathbf{l}$, this always constitutes a solution to

$$C_1^\top\varphi(k_1) + \cdots + C_s^\top\varphi(k_s) = \mathbf{0},$$

as $\varphi(k_j) = 0$ for all $1 \leqslant j \leqslant s$. On the other hand, if we have $\mathbf{k} = \mathbf{k}^* + q^m\mathbf{l}$, where $\mathbf{k}^* \in \mathbb{N}_0^s\setminus\{\mathbf{0}\}$ with all its entries bounded by $q^m - 1$ and $\mathbf{l} \in \mathbb{N}_0^s$, then $\varphi(\mathbf{k}) = \varphi(\mathbf{k}^*)$.

We recall that in Lemma 5.15 we have already defined

$$\mathcal{D}^*_{q^m} := \left\{ (k_1, \ldots, k_s) \in \mathbb{N}_0^s\setminus\{\mathbf{0}\} \colon \ \|\mathbf{k}\|_\infty < q^m \text{ and } \sum_{j=1}^{s} C_j^\top\varphi(k_j) = \mathbf{0} \right\}$$

and simplify in accordance to the above discussion:

$$e^2_{q^m,s}(\mathcal{P}) = \sum_{\mathbf{l}\in\mathbb{N}_0^s\setminus\{0\}} r(\beta, \boldsymbol{\gamma}, q^m\mathbf{l}) + \sum_{\mathbf{k}\in\mathcal{D}^*_{q^m}} \sum_{\mathbf{l}\in\mathbb{N}_0^s} r(\beta, \boldsymbol{\gamma}, \mathbf{k} + q^m\mathbf{l}) = \Sigma_1 + \Sigma_2. \quad (56)$$

From (26) we get that

$$\Sigma_1 = \prod_{j=1}^{s} \left(1 + \gamma_j \frac{\mu(\beta)}{q^{m\beta}}\right) - 1. \tag{57}$$

The simplification and estimation of Σ_2 involves a little more effort. First of all, we fix an arbitrary $j \in [s]$ and consider the jth component of $\mathbf{l}$ in the innermost sum:

$$\begin{aligned}\sum_{l=0}^{\infty} r(\beta, \gamma_j, k_j + q^m l) &= r(\beta, \gamma_j, k_j) + \sum_{l=1}^{\infty} r(\beta, k_j + \gamma_j, q^m l) \\ &= r(\beta, \gamma_j, k_j) + \sum_{l=1}^{\infty} \gamma_j q^{-\beta \lfloor \log_q (k_j + q^m l) \rfloor}.\end{aligned}$$

Since, by definition of $\mathcal{D}^*_{q^m}$, we have that $k_j < q^m$ we obtain

$$\sum_{l=0}^{\infty} r(\beta, \gamma_j, k_j + q^m l) \overset{(26)}{=} r(\beta, \gamma_j, k_j) + \gamma_j \frac{\mu(\beta)}{q^{m\beta}}.$$

Inserting this identity into the definition of Σ_2 yields

$$\begin{aligned}\Sigma_2 &= \sum_{\mathbf{k} \in \mathcal{D}^*_{q^m}} \prod_{j=1}^{s} \left(\sum_{l=0}^{\infty} r(\beta, \gamma_j, k_j + q^m l) \right) \\ &= \sum_{\mathbf{k} \in \mathcal{D}^*_{q^m}} \prod_{j=1}^{s} \left(r(\beta, \gamma_j, k_j) + \gamma_j \frac{\mu(\beta)}{q^{m\beta}} \right).\end{aligned}$$

Now, we can rewrite the above as follows:

$$\begin{aligned}\Sigma_2 &= \sum_{\mathbf{k} \in \mathcal{D}^*_{q^m}} \sum_{\mathrm{u} \subseteq [s]} \left(\prod_{j \in \mathrm{u}} r(\beta, \gamma_j, k_j) \right) \left(\prod_{j \in [s] \setminus \mathrm{u}} \gamma_j \frac{\mu(\beta)}{q^{m\beta}} \right) \\ &= \sum_{\mathbf{k} \in \mathcal{D}^*_{q^m}} r(\beta, \boldsymbol{\gamma}, \mathbf{k}) + \sum_{\mathbf{k} \in \mathcal{D}^*_{q^m}} \sum_{\mathrm{u} \subsetneq [s]} \left(\prod_{j \in \mathrm{u}} r(\beta, \gamma_j, k_j) \right) \left(\prod_{j \in [s] \setminus \mathrm{u}} \gamma_j \frac{\mu(\beta)}{q^{m\beta}} \right) \\ &= \sum_{\mathbf{k} \in \mathcal{D}^*_{q^m}} r(\beta, \boldsymbol{\gamma}, \mathbf{k}) + \sum_{\mathrm{u} \subsetneq [s]} \left(\sum_{\mathbf{k} \in \mathcal{D}^*_{q^m}} \prod_{j \in \mathrm{u}} r(\beta, \gamma_j, k_j) \right) \left(\prod_{j \in [s] \setminus \mathrm{u}} \gamma_j \frac{\mu(\beta)}{q^{m\beta}} \right).\end{aligned} \tag{58}$$

We may, at this point, use Lemma 5.15 to determine an upper bound for the second sum in (58):

$$\begin{aligned}
&\sum_{\mathrm{u}\subsetneq[s]}\left(\sum_{\mathbf{k}\in\mathcal{D}^*_{q^m}}\prod_{j\in\mathrm{u}} r(\beta,\gamma_j,k_j)\right)\left(\prod_{j\in[s]\setminus\mathrm{u}}\gamma_j\frac{\mu(\beta)}{q^{m\beta}}\right)\\
&\overset{(57)}{\leqslant}\frac{1}{q^{m\beta}}\prod_{j=1}^{s}(1+2\gamma_j\mu(\beta))-\Sigma_1\\
&=\frac{1}{q^{m\beta}}\Big(1+\sum_{\varnothing\neq\mathrm{u}\subseteq[s]}\prod_{j\in\mathrm{u}}(2\mu(\beta)\gamma_j)\Big)-\Sigma_1\\
&\leqslant\frac{1}{q^{m\beta}}\left(1+\sum_{\varnothing\neq\mathrm{u}\subseteq[s]}\frac{q^\beta-1}{q^\beta}q^{\beta t_{\mathrm{u}}}\prod_{j\in\mathrm{u}}\left(q^{\beta+1}(m+2)\mu(\beta)\gamma_j\right)\right)-\Sigma_1. \qquad (59)
\end{aligned}$$

In the following we draw our attention to the first sum in (58), i.e.

$$\sum_{\mathbf{k}\in\mathcal{D}^*_{q^m}} r(\beta,\boldsymbol{\gamma},\mathbf{k})=\sum_{\substack{k_1,\dots,k_s=0,\ \|\mathbf{k}\|_\infty\neq0\\ C_1^\top\varphi(k_1)+\dots+C_s^\top\varphi(k_s)=\mathbf{0}}}^{q^m-1} r(\beta,\gamma_1,k_1)\cdots r(\beta,\gamma_s,k_s),$$

where $\mathbf{k}=(k_1,\dots,k_s)$. Since $r(\beta,\gamma_j,0)=1$ for any $1\leqslant j\leqslant s$ and $\varphi(0)=\mathbf{0}$, we may rearrange this sum in the following way, according to for which indices $1\leqslant j\leqslant s$ we have $k_j=0$:

$$\sum_{\mathbf{k}\in\mathcal{D}^*_{q^m}} r(\beta,\boldsymbol{\gamma},\mathbf{k})=\sum_{\substack{\varnothing\neq\mathrm{u}\subseteq[s]\\ \mathrm{u}=\{u_1,\dots,u_e\}}}\ \sum_{\substack{k_{u_1},\dots,k_{u_e}=1\\ C_{u_1}^\top\varphi(k_{u_1})+\dots+C_{u_e}^\top\varphi(k_{u_e})=\mathbf{0}}}^{q^m-1}\ \prod_{j\in\mathrm{u}} r(\beta,\gamma_j,k_j).$$

For an arbitrary, non-empty subset $\mathrm{u}=\{u_1,\dots,u_e\}$ of $[s]$ we define $\mathcal{B}(\mathrm{u})$ as in Lemma 5.17 and exploit the result stated therein to find that

$$\begin{aligned}
\sum_{\mathbf{k}\in\mathcal{D}^*_{q^m}} r(\beta,\boldsymbol{\gamma},\mathbf{k})&=\sum_{\varnothing\neq\mathrm{u}\subseteq[s]}\mathcal{B}(\mathrm{u})\\
&\leqslant\sum_{\varnothing\neq\mathrm{u}\subseteq[s]}\left(\frac{q-1}{q^{\beta-1}-1}\right)^{|\mathrm{u}|}\frac{2(m-t_{\mathrm{u}}+2)^{|\mathrm{u}|-1}}{q^{\beta(m-t_{\mathrm{u}}+1-2|\mathrm{u}|)}}\prod_{j\in\mathrm{u}}\gamma_j.
\end{aligned}$$

Since we have

$$\frac{q-1}{q^{\beta-1}-1}=q^{1-\beta}\frac{q^\beta(q-1)}{q^\beta-q}=q^{1-\beta}\mu(\beta)$$

and

$$2(m-t_{\mathrm{u}}+2)^{|\mathrm{u}|-1}\leqslant(m-t_{\mathrm{u}}+2)^{|\mathrm{u}|}\leqslant(m+2)^{|\mathrm{u}|},$$

it follows that

$$\sum_{\mathbf{k}\in\mathcal{D}^*_{q^m}} r(\beta,\boldsymbol{\gamma},\mathbf{k}) \leqslant \sum_{\varnothing\neq\mathrm{u}\subseteq[s]} q^{-\beta(m-t_{\mathrm{u}}+1-2|\mathrm{u}|)} \prod_{j\in\mathrm{u}} \left(q^{1-\beta}\mu(\beta)(m+2)\gamma_j\right)$$
$$= \frac{1}{q^{(m+1)\beta}} \sum_{\varnothing\neq\mathrm{u}\subseteq[s]} q^{\beta t_{\mathrm{u}}} \prod_{j\in\mathrm{u}} \left(q^{\beta+1}(m+2)\mu(\beta)\gamma_j\right). \tag{60}$$

As a summary, we obtain

$$\begin{aligned}
e^2_{q^m,s}(\mathcal{P}) &\overset{(56)}{=} \Sigma_1+\Sigma_2\\
&\overset{(58),(59),(60)}{\leqslant} \frac{1}{q^{(m+1)\beta}} \sum_{\varnothing\neq\mathrm{u}\subseteq[s]} q^{\beta t_{\mathrm{u}}} \prod_{j\in\mathrm{u}} \left(q^{\beta+1}(m+2)\mu(\beta)\gamma_j\right)\\
&\qquad + \frac{1}{q^{m\beta}} \left(1+\sum_{\varnothing\neq\mathrm{u}\subseteq[s]} \frac{q^\beta-1}{q^\beta} q^{\beta t_{\mathrm{u}}} \prod_{j\in\mathrm{u}} \left(q^{\beta+1}(m+2)\mu(\beta)\gamma_j\right)\right)\\
&= \frac{1}{q^{m\beta}} \left(1+\sum_{\varnothing\neq\mathrm{u}\subseteq[s]} q^{\beta t_{\mathrm{u}}} \prod_{j\in\mathrm{u}} \left(q^{\beta+1}(m+2)\mu(\beta)\gamma_j\right)\right),
\end{aligned}$$

which is exactly what we wanted to show. □

Finally, we can reap the benefits of the hard work we have had proving the preceding lemmas, as the task of providing an upper bound for the worst-case error for digital nets constructed by Niederreiter's method is a perfectly easy one now.

Theorem 5.19. *Let $\mathcal{P}$ be a digital (t, m, s)-net over $\mathbb{F}_q$ constructed by Niederreiter's method as introduced in the beginning of this section by using the first s polynomials $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ from a list of monic and irreducible polynomials over $\mathbb{F}_q$ ordered by their degree in a non-decreasing manner. Then, the worst-case error for integration in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ using $\mathcal{P}$ as sample points is bounded by*

$$e^2_{q^m,s}(\mathcal{P}) \leqslant \frac{1}{q^{m\beta}} \prod_{j=1}^{s} \left(1+q^{2\beta+1}\left(j\log_q(j+q)\right)^\beta (m+2)\mu(\beta)\gamma_j\right).$$

Moreover, if we have

$$\sum_{j=1}^{\infty} (j\log j)^\beta \gamma_j < \infty,$$

then

$$e^2_{q^m,s}(\mathcal{P}) \leqslant c_{\delta,q,\beta,\gamma} q^{m(\delta-\beta)}$$

for any $\delta > 0$, where $c_{\delta,q,\beta,\gamma}$ is a positive constant which is independent of m and s.

(cf. [3, Lemma 5])

Proof. First of all, we have to make sure that all the generating matrices involved are regular. According to [3, p. 287] this can be achieved by a slight modification of the generating matrices, which, however, has neither an effect on the quality parameter of the digital net itself, nor on those of its projections.

In what follows, we consider the main steps of the proof of [3, Lemma 5]. From [19, p. 7] we learn that for any non-empty $\mathrm{u} \subseteq [s]$ we have

$$t_{\mathrm{u}} = \sum_{j \in \mathrm{u}} (\deg(\mathfrak{p}_j) - 1),$$

where $\mathfrak{p}_j$ denotes the jth monic irreducible polynomial used in the construction scheme. The only thing left to do is to determine an upper bound for the degree of $\mathfrak{p}_j$, $1 \leqslant j \leqslant s$. To this end, we mention that, by assumption, these polynomials are the first s monic and irreducible polynomials listed according to non-decreasing degree and cite [19, Lemma 2], where it is shown that

$$\deg(\mathfrak{p}_j) \leqslant \log_q j + \log_q \log_q(j+q) + 2$$

for all $1 \leqslant j \leqslant s$. All in all this implies

$$q^{t_{\mathrm{u}}} \leqslant \prod_{j \in \mathrm{u}} \left(qj \log_q(j+q)\right).$$

We use this bound on Lemma 5.18 and obtain

$$\begin{aligned} q^{m\beta} e^2_{q^m,s}(\mathcal{P}) &\leqslant 1 + \sum_{\emptyset \neq \mathrm{u} \subseteq [s]} \prod_{j \in \mathrm{u}} \left(q^{2\beta+1} \left(j \log_q(j+q)\right)^{\beta} (m+2)\mu(\beta)\gamma_j \right) \\ &= \prod_{j=1}^{s} \left(1 + q^{2\beta+1} \left(j \log_q(j+q)\right)^{\beta} (m+2)\mu(\beta)\gamma_j \right), \end{aligned}$$

which proves the first assertion.

For the second part we proceed as it was done in [10, Lemma 3]. To this end we define

$$\tilde{\gamma}_j := q^{2\beta+1}\mu(\beta)\left(j\log_q(j+q)\right)^{\beta}\gamma_j$$

for all integers $j \geqslant 1$ as well as

$$\sigma_k := \sum_{j=k+1}^{\infty}\tilde{\gamma}_j$$

for all $k \geqslant 0$. For any such k we obtain the following inequality:

$$\begin{aligned}
&\log\left(\prod_{j=1}^{s}(1+(m+2)\tilde{\gamma}_j)\right)\\
&\leqslant \sum_{j=1}^{\infty}\log(1+(m+2)\tilde{\gamma}_j)\\
&\leqslant \sum_{j=1}^{k}\log\left(1+\sigma_k^{-1}+(m+2)\tilde{\gamma}_j\right)+\sum_{j=k+1}^{\infty}\log(1+(m+2)\tilde{\gamma}_j)\\
&=\sum_{j=1}^{k}\log\left(\left(1+\sigma_k^{-1}\right)\left(1+\frac{(m+2)\tilde{\gamma}_j}{1+\sigma_k^{-1}}\right)\right)+\sum_{j=k+1}^{\infty}\log(1+(m+2)\tilde{\gamma}_j)\\
&=k\log\left(1+\sigma_k^{-1}\right)+\sum_{j=1}^{k}\log\left(1+\frac{(m+2)\tilde{\gamma}_j}{1+\sigma_k^{-1}}\right)+\sum_{j=k+1}^{\infty}\log(1+(m+2)\tilde{\gamma}_j)\\
&\leqslant k\log\left(1+\sigma_k^{-1}\right)+(m+2)\sigma_k\sum_{j=1}^{k}\tilde{\gamma}_j+(m+2)\sum_{j=k+1}^{\infty}\tilde{\gamma}_j\\
&\leqslant k\log\left(1+\sigma_k^{-1}\right)+(m+2)\sigma_k\sigma_0+(m+2)\sigma_k\\
&=k\log\left(1+\sigma_k^{-1}\right)+(m+2)\sigma_k(\sigma_0+1).
\end{aligned}$$

As

$$\sum_{j=1}^{\infty}(j\log j)^{\beta}\gamma_j<\infty$$

by assumption, it follows that for any $\delta>0$ and some sufficiently large k_δ we have

$$\sigma_{k_\delta}(\sigma_0+1)<\delta\log q.$$

Consequently, we obtain from the first claim of this theorem in combination with the above discussion that

$$
\begin{aligned}
e^2_{q^m,s}(\mathcal{P}) &\leqslant \frac{1}{q^{m\beta}} \prod_{j=1}^{s} \left(1 + (m+2)\tilde{\gamma}_j\right) \\
&\leqslant \frac{1}{q^{m\beta}} \left(1 + \sigma_{k_\delta}^{-1}\right)^{k_\delta} \exp\left((m+2)\sigma_{k_\delta}(\sigma_0 + 1)\right) \\
&\leqslant q^{2\delta} \left(1 + \sigma_{k_\delta}^{-1}\right)^{k_\delta} q^{m(\delta-\beta)}
\end{aligned}
$$

and the result follows by setting

$$
c_{\delta,q,\beta,\boldsymbol{\gamma}} := q^{2\delta} \left(1 + \sigma_{k_\delta}^{-1}\right)^{k_\delta}.
$$

□

Here we easily see the important property that, under certain conditions on the sequence of weights $(\gamma_j)_{j\in\mathbb{N}}$, the error bound does not depend on s. Hence, it follows that strong tractability of integration in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ may be exploited.

5.3 The construction method by Sobol'

Here, the generating matrices are constructed in a very similar fashion, compared to the method proposed by Niederreiter (see Section 5.2). Hence, it rather does not come as a surprise that the error estimation for digital nets constructed by Sobol's method is similar to the result we had in Theorem 5.19. The construction principle can be seen in the paragraphs below or, alternatively, in [5, Section 8.1.3], for instance.

This method only considers the case $q = 2$. We set $\mathfrak{p}_1(x) = x$ and choose $s-1$ primitive polynomials[1] over $\mathbb{F}_2$ and order them according to their degree in an increasing manner, say

$$
\deg(\mathfrak{p}_2) \leqslant \deg(\mathfrak{p}_3) \leqslant \ldots \leqslant \deg(\mathfrak{p}_s).
$$

Furthermore, we define $e_j = \deg(\mathfrak{p}_j)$ and choose polynomials $y_{i,j,k}(x)$, where $1 \leqslant j \leqslant s$, $1 \leqslant i \leqslant m$ and $0 \leqslant k < e_j$. The only restriction we need to put on these polynomials is that for every $1 \leqslant j \leqslant s$ the sets $\{y_{i,j,k}(x)\colon\ 0 \leqslant k < e_j\}$ are linearly independent over $\mathbb{F}_2$, with the arithmetics taken mod $\mathfrak{p}_j(x)$.

[1] A primitive polynomial over a finite field $\mathbb{F}$ is a monic irreducible polynomial whose roots are generators of the multiplicative group $\mathbb{F}\backslash\{0\}$, (cf. [9, p. 639]).

Then, analogously to the construction method by Niederreiter, we consider the formal Laurent series expansion

$$\frac{y_{i,j,k}(x)}{p_j(x)^i} = \sum_{r=0}^{\infty} a^{(j)}(i,k,r)x^{-r-1},$$

where the possible values of i, j and k are the same as above. The entries of the jth generating matrice are now defined as follows:

$$c_{i,r}^{(j)} = a^{(j)}(Q+1,k,r),$$

where $1 \leqslant i \leqslant m$, $0 \leqslant r \leqslant m-1$ and where the integers Q and $0 \leqslant k < e_j$ satisfy

$$i - 1 = Qe_j + k.$$

For any digital (t,m,s)-net constructed by the above scheme we obtain the following result:

Theorem 5.20. *Let $\mathcal{P}$ be a digital (t,m,s)-net over $\mathbb{F}_2$ constructed by Sobol's method. Then, the worst-case error for integration in $\mathscr{H}_{\mathbb{F}_2,\varphi_1 \mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ employing the point set $\mathcal{P}$ satisfies*

$$e^2_{2^m,s}(\mathcal{P}) \leqslant \frac{1}{2^{m\beta}} \prod_{j=1}^{s} \left(2^{\beta c+1} \left(j \log_2(j+1) \log_2 \log_2(j+3) \right)^{\beta} (m+2)\mu(\beta)\gamma_j \right),$$

where c is a constant independent of all parameters.

Furthermore, if

$$\sum_{j=1}^{\infty} \left(j \log j \log \log j \right)^{\beta} \gamma_j < \infty,$$

then the worst-case error can be bounded by

$$e^2_{2^m,s}(\mathcal{P}) \leqslant c_{\delta,\beta,\boldsymbol{\gamma}} 2^{m(\delta-\beta)}$$

for any $\delta > 0$, where the constant $c_{\delta,\beta,\boldsymbol{\gamma}}$ is independent of s.

(cf. [3, Lemma 5])

Proof. We closely follow the proof of Theorem 5.19, so, as a matter of fact, the proof of [3, Lemma 5]. First, we refer to [3, p. 287] to find that the generating matrices can be made regular without altering important parameters of the digital net or its projection and hence Lemma 5.18 can be applied.

For the determination of the quality paramteters t_{u}, $\varnothing \neq \mathrm{u} \subseteq [s]$, we cite [19, p. 835], which, again, states that

$$t_{\mathrm{u}} = \sum_{j \in \mathrm{u}} (\deg(\mathfrak{p}_j) - 1)\,.$$

We recall that here, $\mathfrak{p}_j$ denotes the jth element in a previously chosen set of s primitive polynomials which were sorted by degree in an increasing order. According to [19, p. 836] we have

$$\deg(\mathfrak{p}_j) \leqslant \log_2 j + \log_2 \log_2(j+1) + \log_2 \log_2 \log_2(j+3) + c,$$

where c is a constant indepent of j and s. All in all this yields (note that here, $q = 2$ by definition)

$$2^{t_{\mathrm{u}}} \leqslant \prod_{j \in \mathrm{u}} \left(2^{c-1} j \log_2(j+1) \log_2 \log_2(j+3)\right)$$

and after using this inequality on Lemma 5.18 the first part of this theorem immediately follows.

For the second part we proceed in a similar fashion as in the proof of [10, Lemma 3] and define a new sequence of weights

$$\tilde{\gamma}_j := 2^{\beta c+1} \left(j \log_2(j+1) \log_2 \log_2(j+3)\right)^{\beta} \mu(\beta) \gamma_j,$$

$j \in \mathbb{N}$. Due to the assumption

$$\sum_{j=1}^{\infty} \left(j \log j \log \log j\right)^{\beta} \gamma_j < \infty$$

we have that

$$\sum_{j=1}^{\infty} \tilde{\gamma}_j < \infty.$$

Hence, it makes sense to define

$$\sigma_k := \sum_{j=k+1}^{\infty} \tilde{\gamma}_j.$$

Once again, for a fixed $\delta > 0$ we choose k_δ such that

$$\sigma_{k_\delta}(\sigma_0 + 1) < \delta \log 2.$$

Following the respective steps in the proof of Theorem 5.19 we get

$$\prod_{j=1}^{\infty} (1 + \tilde{\gamma}_j(m+1)) \leqslant \left(1 + \sigma_{k_\delta}^{-1}\right)^{k_\delta} \exp\left((m+2)\sigma_{k_\delta}(\sigma_0 + 1)\right)$$

and finish the proof by observing that

$$\begin{aligned} e^2_{2^m,s}(\mathcal{P}) &\leqslant \frac{1}{2^{m\beta}} \prod_{j=1}^{\infty} (1 + \tilde{\gamma}_j(m+1)) \\ &\leqslant 2^{2\delta} \left(1 + \sigma_{k_\delta}^{-1}\right)^{k_\delta} 2^{m(\delta - \beta)}. \end{aligned}$$

□

Again, we close this section by briefly commenting on the last two construction methods, i.e. Niederreiter's and Sobol's approach. We immediately notice that the constants in the respective error estimations may increase vastly as δ approaches zero. Nevertheless, from these error bounds we see that the methods may exploit strong tractability if applicable, since we have managed to bound the worst-case error independently of the dimension s.

Moreover, if we compare the results we have shown for the CBC construction (see Corollary 5.10) to those which we have had for Niederreiter's method and Sobol's method (see Theorems 5.19 and 5.20), we notice that for the latter two we had to impose stronger conditions on the sequence $(\gamma_j)_{j\in\mathbb{N}}$ in order to have the possibility to achieve strong tractability than for the CBC construction method.

6 Concluding remarks

After having laid the necessary groundwork, that is, in particular, to establish that the space of generalized Walsh series $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ is a reproducing kernel Hilbert space, we used this fact to find an explicit formula for the worst-case error, which is computable at a cost of $\mathcal{O}\left(n^2 s\right)$ operations (see Remark 4.8) and this effort can be even reduced to $\mathcal{O}(ns)$ operations when employing digital nets as sample points (see Theorem 4.12).

Subsequently, we used averaging techniques to guarantee the existence of a digital (t, m, s)-net such that the worst-case error is bounded by

$$e_{q^m,s} \leqslant c_{s,\boldsymbol{\gamma},\lambda,\beta} q^{-\frac{m}{2\lambda}},$$

where $c_{s,\boldsymbol{\gamma},\lambda,\beta}$ denotes a constant depending on the quantities given in its index and where $\lambda \in (1/\beta, 1]$ (see Theorem 4.15), from which we were able to deduce that there exists a digitial (t, m, s)-net such that (strong) tractability can be achieved, whenever the sequence of weights $\boldsymbol{\gamma}$ fulfills certain conditions (see Corollaries 4.16 and 4.18). Moreover, we even managed to show that these conditions (with $\lambda = 1$) are also necessary for integration in $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$ to be (strongly) QMC-tractable (see Corollary 4.21).

These results, however, left us partly unsatisfied, as none of their proofs was of a constructive nature. To this end we included a fifth section, where four construction algorithms for digital (t, m, s)-nets were presented and their performance, with respect to their error behavior, was investigated. These algorithms included:

1. The component-by-component (CBC) construction (see Algorithm 5.8),
2. a Korobov type construction (see Algorithm 5.11),
3. the construction method by Niederreiter (see the beginning of Section 5.2) and
4. the construction method by Sobol‘ (see the beginning of Section 5.3).

As it was mentioned before in Remark 5.12, the Korobove type construction is approximately s times faster than the CBC construction. The worst-case error, however, satisfies an error bound which is, roughly speaking, $\sqrt{s}$ times smaller for the latter (see Theorems 5.9 and 5.13). This also lead to the fact that, again, under certain conditions on $\boldsymbol{\gamma}$, we may achieve strong tractability in case of the CBC construction, whereas with the Korobov type construction the error bound still depends polynomially on s (see Corollaries 5.10 and 5.14).

Finally, after having taken a huge effort to find an appropriate bound for the worst-case error for point sets constructed by the Niederreiter method and the Sobol‘ method respectively, we were again able to find conditions we have to put on the weights $\boldsymbol{\gamma}$ such that strong tractability can be made use of (see Theorems 5.19 and 5.20), which are, however, still stronger than those we have to impose on the weights to obtain a similar result for the CBC construction method.

A Bibliography

[1] N. Aronszajn. *Theory of reproducing kernels*, Trans. Amer. Math. Soc. Vol. 68: pp.337-404, 1950.

[2] L. L. Cristea, J. Dick, F. Pillichshammer. *On the mean square weighted $\mathcal{L}_2$ discrepancy of randomized digital nets in prime base*, Journal of Complexity 22: pp. 605-629, 2006.

[3] J. Dick, P. Kritzer, F. Y. Kuo. *Approximation of functions using digital nets*, in "Monte Carlo and Quasi-Monte Carlo Methods 2006": pp. 275-297, Springer-Verlag, Berlin, 2008.

[4] J. Dick, F. Y. Kuo, F. Pillichshammer, I. H. Sloan. *Construction algorithms for polynomial lattice rules for multivariate integration*, Mathematics of Computation Vol. 74: pp. 1895-1921, 2005.

[5] J. Dick, F. Pillichshammer. *Digital nets and sequences. Discrepancy theory and quasi-Monte Carlo integration*, Cambridge University Press, Cambridge 2010.

[6] J. Dick, F. Pillichshammer. *Multivariate integration in weighted Hilbert spaces based on Walsh functions and weighted Sobolev spaces*, Journal of Complexity 21: pp. 149-195, 2005.

[7] J. Dick, F. Pillichshammer. *On the mean square weighted $\mathcal{L}_2$ discrepancy of randomized digital (t, m, s)-nets over $\mathbb{Z}_2$*, Acta Arithmetica 117: pp. 371-403, 2005.

[8] J. Dick, I. H. Sloan, X. Wang, H. Woźniakowski. *Good lattice rules in weighted Korobov spaces with general weights*, Numerische Mathematik 103(1): pp. 63-97, Springer-Verlag, 2006.

[9] T. Hansen, G. L. Mullen. *Primitive polynomials over finite fields*, Mathematics of Computation Vol. 59: pp. 639-643, 1992.

[10] F. J. Hickernell, H. Niederreiter. *The existence of good extensible rank-1 lattices*, Journal of Complexity 19: pp. 286-300, 2003.

[11] R. Lidl, H. Niederreiter. *Introduction to finite fields and their applications*, Cambridge University Press, Cambridge 1994.

[12] H. Niederreiter. *Low-discrepancy point sets obtained by digital constructions over finite fields*, Czechoslovak Mathematical Journal 42: pp. 143-166, 1992.

[13] H. Niederreiter. *Point sets and sequences with small discrepancy*, Monatshefte für Mathematik 104(4): pp. 273-337, 1987.

[14] H. Niederreiter. *Random number generation and quasi-Monte Carlo methods*, Number 63 in CBMS-NSF Series in Applied Mathematics, SIAM, Philadelphia, 1992.

[15] G. Pirsic. *Schnell konvergierende Walshreihen über Gruppen*, Masters thesis, University of Salzburg, 1995. (Available at http://www.ricam.oeaw.ac.at/people/page/pirsic/)

[16] G. Pirsic, J. Dick, F. Pillichshammer. *Cyclic digital nets, hyperplane nets, and multivariate integration in Sobolev spaces*, SIAM J. Numer. Anal. 44-1: pp. 385-411, 2006.

[17] G. Pirsic, F. Pillichshammer. *Extensible hyperplane nets*, Finite Fields and Their Applications 17: pp. 407-423, 2011.

[18] I. H. Sloan, H. Woźniakowski. *Tractability of multivariate integration for weighted Korobov classes*, Journal of Complexity 17: pp. 697-721, 2001.

[19] X. Wang. *Strong tractability of multivariate integration using quasi-Monte Carlo algorithms*, Mathematics of Computation Vol. 72: pp. 823-838, 2003.

B List of abbreviations

CBC	component-by-component
cf.	confer
e.g.	exempli gratia, for example
f./ff.	and the following page/ and the following pages
i.e.	id est, that is
iff	if and only if
p./pp.	page/pages
prop.	proposition
QMC	quasi-Monte Carlo
rem.	remark
thm.	theorem
w.l.o.g	without loss of generality

C List of figures

D List of globally used symbols

Numbers, vectors and matrices

As a general guideline we mention that numbers and matrices are printed in regular letters (e.g. x stands for some number, C for some matrix,...), while vectors are printed in bold letters (e.g. $\mathbf{x}$). Moreover, for (vectors of) polynomials (bold) fraktur letters are used (e.g. $\mathfrak{p}, \boldsymbol{\mathfrak{p}}, \boldsymbol{\mathfrak{k}}, \ldots$).

r	positive integer
p	prime number
q	prime power, $q := p^r$
s	positive integer, usually denotes the dimension
β	real number, $\beta > 1$
i	imaginary unit, $\mathrm{i}^2 = -1$
0, $\mathbf{0}$	neutral element of addition (the corresponding space should be clear from the context)
γ	positive weight
$\boldsymbol{\gamma}$	$\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots)$ sequence of non-increasing positive weights
$\boldsymbol{\gamma}^\lambda$	$\boldsymbol{\gamma}^\lambda = (\gamma_1^\lambda, \gamma_2^\lambda, \ldots)$, where λ is real
δ_{ij}	Kronecker delta, i.e. $\delta_{ij} \in \{0, 1\}$ and $\delta_{ij} = 1$ iff $i = j$ where i, j are positive integers.
$r(\beta, \gamma, k)$	see Equation (4), where k is a positive integer
$r(\beta, \boldsymbol{\gamma}, \mathbf{k})$	see Definition 2.22, where $\mathbf{k}$ a vector of positive integers
$\mu(\beta)$	see Lemma 2.16
$\hat{f}(k)$, $\hat{f}(\mathbf{k})$	kth (or $\mathbf{k}$th) Walsh-Fourier coefficient of a Walsh series $f \in \mathscr{H}_{\mathrm{wal},\beta,\gamma}$ (or $f \in \mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$), where k is a non-negative integer (or $\mathbf{k}$ is a vector of non-negative integers), see Definition 2.11 (or Definition 2.22)

Sets and set-related operations

X	arbitrary set
X^s	s-fold cartesian product of a set X
$\lvert X\rvert$, $\#X$	cardinality of a set X
$\lambda(X)$	Lebesgue measure of a set X
$\varnothing$	empty set
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathbb{N}$	non-negative integers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$[d]$	the set $\{1, 2, \ldots, d\}$, $d \in \mathbb{N}$
$\mathbb{Z}$	integers
$\mathbb{Z}_b$	residue class ring modulo b, $b \in \mathbb{N}$; usually identified with the least residue system modulo b, i.e. $\{0, 1, \ldots, b-1\}$
$\mathbb{F}$, $\mathbb{F}_q$	finite field (with q elements, where q is a prime power)
$\mathbb{F}^{m\times m}$	set of all $m \times m$ matrices over $\mathbb{F}$, $m \in \mathbb{N}$
$\mathbb{F}[x]$	field of polynomials over $\mathbb{F}$
$\mathbb{F}((x^{-1}))$	field of formal Laurent series over $\mathbb{F}$, see beginning of Section 5
$G_{q,m}$	set of non-zero polynomials over $\mathbb{F}_q$ with degree less than $m \in \mathbb{N}$
$\mathcal{D}$, $\mathcal{D}^*$	dual net (or dual net without zero), see Definition 4.11
$\mathcal{D}_{\mathfrak{p},\mathbf{q}}$, $\mathcal{D}^*_{\mathfrak{p},\mathbf{q}}$	polynomial related version of dual net (dual net without zero) see Lemma 5.7
$\mathcal{S}_{\mathfrak{p}}(\mathbf{q})$	polynomial lattice point set, see Definition 5.1
$\mathscr{H}$	Hilbert space (general case)
$\mathscr{H}_{\mathrm{wal},\beta,\gamma}$	one-dimensional Hilbert space of generalized Walsh series, see Corollary 2.15
$\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$	s-dimensional analogon of $\mathscr{H}_{\mathrm{wal},\beta,\gamma}$, $s \in \mathbb{N}$, see Definition 2.22
$L_2(X)$	Hilbert space of square-integrable functions defined on a set X
$\mathscr{H}_1 \otimes \mathscr{H}_2$	tensor product of two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$

Special operations, maps, functions and operators

$\overline{z}$	complex conjugate of $z \in \mathbb{C}$
$\lvert z\rvert$	absolute value of $z \in \mathbb{C}$
$\lfloor x \rfloor$	floor function of $x \in \mathbb{R}$
$\lVert\mathbf{k}\rVert_\infty$	$\max\{k_1, \dots, k_d\}$ for $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}_0^d$, $d \in \mathbb{N}$
$\mathbf{a}^\top$, $C^\top$	transpose of a vector $\mathbf{a}$ or of a matrix C
$\deg(\mathfrak{p})$	degree of a polynomial $\mathfrak{p}$
$v_s(\mathfrak{q})$	see paragraph before Algorithm 5.11
tr_m	truncation operator, $m \in \mathbb{N}$; see paragraph after Theorem 5.6
$\mathbf{a} \cdot \mathbf{b}$	Euklidean product of two vectors $\mathbf{a}$ and $\mathbf{b}$
$\boldsymbol{\mathfrak{p}} \cdot \boldsymbol{\mathfrak{q}}$	$\sum_{j=1}^s \mathfrak{p}_j \mathfrak{q}_j$, where $\boldsymbol{\mathfrak{p}} = (\mathfrak{p}_1, \dots, \mathfrak{p}_s)$, $\boldsymbol{\mathfrak{q}} = (\mathfrak{q}_1, \dots, \mathfrak{q}_s) \in \mathbb{F}_q^s$
$\oplus$, $\ominus$	binary, digitwise operation, see Definition 2.4
φ_1	arbitrary bijection from $\mathbb{Z}_q$ to $\mathbb{F}_q$ with $\varphi_1(0) = 0$
φ	extension of φ_1, $\varphi\colon \mathbb{Z}_{q^m} \to (\mathbb{F}_q^m)^\top$, see Lemma 3.8; sometimes also its extension $\varphi\colon \mathbb{N}_0 \to (\mathbb{F}_q^m)^\top$, see paragraph before Definition 4.11
ψ	isomorphism between the additive groups $\mathbb{F}_q$ and $\mathbb{Z}_p^r$, sometimes also its extension, see Figure 2
η	the map $\psi \circ \varphi_1$, sometimes also concatenation of the respective extensions $\psi \circ \varphi$, see Figure 2
$\exp(z)$	exponential function e^z, $z \in \mathbb{C}$
$\log, \log_b$	natural logarithm (or logarithm to base b, $b > 1$)
wal_k, $\mathrm{wal}_{\mathbf{k}}$	kth (or $\mathbf{k}$th) generalized Walsh function over $\mathbb{F}_q$, see Definition 2.3
K	reproducing kernel of a Hilbert space $\mathscr{H}$ (general case), see Definition 2.1
$K_{\mathrm{wal},\beta,\gamma}$	reproducing kernel of $\mathscr{H}_{\mathrm{wal},\beta,\gamma}$, see Theorem 2.17
$K_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$	reproducing kernel of $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$, see Theorem 2.23
$\langle\cdot,\cdot\rangle$	inner product on a Hilbert space $\mathscr{H}$ (general case)
$\langle\cdot,\cdot\rangle_{\mathrm{wal},\gamma}$	inner product on $\mathscr{H}_{\mathrm{wal},\beta,\gamma}$, see paragraph before Lemma 2.12
$\langle\cdot,\cdot\rangle_{\mathrm{wal},s,\boldsymbol{\gamma}}$	inner product on $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$, see Definition 2.22
$\lVert\cdot\rVert$	norm (general case); usually induced norm $\lVert\cdot\rVert := \sqrt{\langle\cdot,\cdot\rangle}$
$\lVert\cdot\rVert_{\mathrm{wal},\gamma}$	induced norm on $\mathscr{H}_{\mathrm{wal},\beta,\gamma}$
$\lVert\cdot\rVert_{\mathrm{wal},s,\boldsymbol{\gamma}}$	induced norm on $\mathscr{H}_{\mathrm{wal},s,\beta,\boldsymbol{\gamma}}$
$Q_{n,s}(f)$	quasi-Monte Carlo-rule for a function f, see the beginning of Section 4
$I_s(f)$	integral operator, see the beginning of Section 4
$e_{n,s}, e(Q_{n,s})$	worst-case error, see Definition 4.4; might take different arguments, depending on what is intended to be emphasized
$n_{\min}(\epsilon, s)$	information complexity, see Definition 4.5
$\mathcal{O}(f(x))$	big O of $f(x)$ in the sense of Landau notation, i.e. $g(x) = \mathcal{O}(f(x))$ iff f is an asymptotical upper bound for g (here for $x \to \infty$)